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**Autor:** Henry, Philippe / Wanner, Gerhard
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**Elemente der Mathematik**

# Zigzags with Bürgi, Bernoulli, Euler and the Seidel–Entringer–Arnol'd triangle

Philippe Henry and Gerhard Wanner

Philippe Henry completed his PhD at the Ecole Polytechnique fédérale de Lausanne under the supervision of Nicolas Monod in 2010. His main interests include group theory and history of geometry.

Gerhard Wanner's Thesis was completed 1965 in Innsbruck under the supervision of Wolfgang Gröbner. His main interest is since then numerics of differential equations.

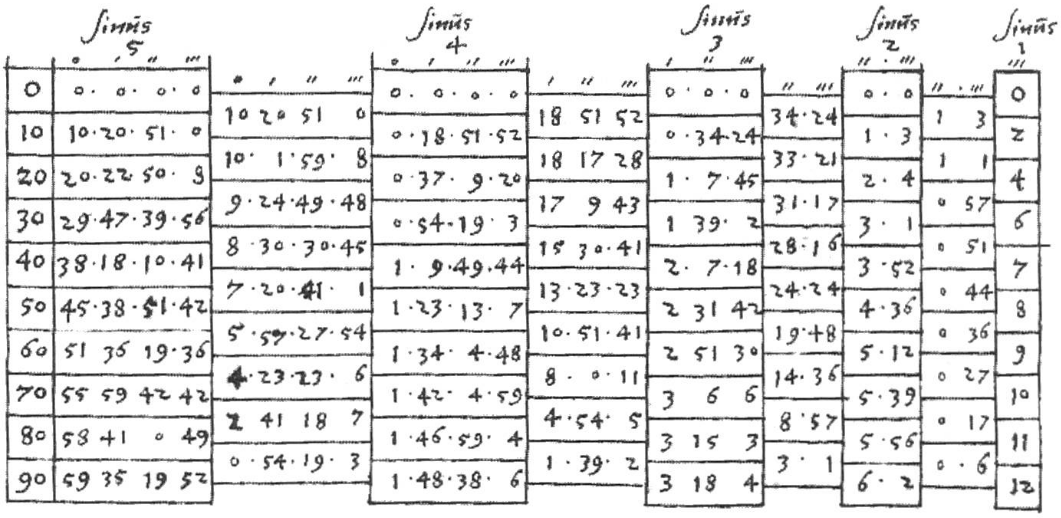
## 1 Jost Bürgi's *Artificium* of 1586

> "For many hundreds of years, up to now, our ancestors have been using this method because they were not able to invent a better one. However, this method is uncertain and dilapidated as well as cumbersome and laborious. Therefore we want to perform this in a different, better, more correct, easier and more cheerful way. And we want to point out now how all sines can be found without the troublesome inscription [of polygons] (...)."
>
> (Jost Bürgi, *Fundamentum Astronomiæ*, fol. 34r-v, [14, p. 140])

A spectacular discovery of a lost manuscript of Jost Bürgi (1552–1632), entitled *Fundamentum Astronomiæ*, was made in the University Library of Wrocław by Menso Folkerts

In diesem Beitrag werden mannigfache Zusammenhänge aufgezeigt zwischen dem kürzlich wiederentdeckten *Artificium* von Jost Bürgi, den *iterierten Evolventen* von Johann Bernoulli und der *Kombinatorik alternierender Permutationen* von Désiré André. Dabei ergeben sich in ganz natürlicher Weise einfache und anschauliche Beweise. Die Sinusfunktion, die Eulerschen und Bernoullischen Zahlen spielen dabei eine zentrale Rolle und erlauben eine geometrische Herleitung der Reihen für $\tan x$ und $\sec x$, welche die exponentiell erzeugende Funktion der Eulerschen Zickzackzahlen darstellen. Diese letzteren kommen bereits bei Johann Bernoulli vor. Das Seidel–Entringer–Arnol'd Dreieck lässt sich auf alternierende Injektionen erweitern, was einen Beweis des Boustrophedon Theorems erlaubt sowie eine zweite geometrische Herleitung der obigen Reihen.

in 2013[1]. This manuscript contains a forgotten iterative method which is also discussed in the papers Waldvogel [30], Folkerts, Launert, Thom [14] and Nicollier [25]. Bürgi's work was crowned by his *Canonis Sinuum*, an impressive table of $90 \cdot 60 = 5400$ sine values with a precision of 5 hexadecimal digits (approximately 9 decimal digits) obtained using trigonometric identities and interpolations. This table fills 36 pages of the *Fundamentum Astronomiæ* (fol. 46v–64r), which was never published and rediscovered only four centuries later.



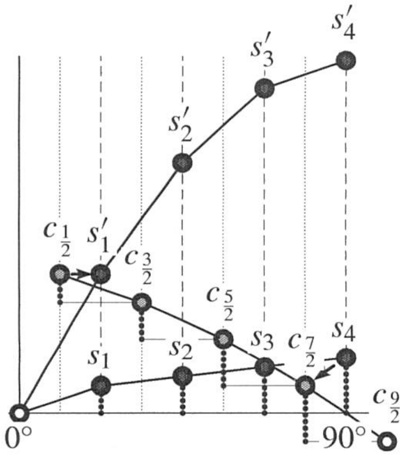
| | Sinus 1 | | Sinus 2 | | Sinus 3 | | Sinus 4 | | Sinus 5 |
|---|---|---|---|---|---|---|---|---|---|
| 90° | 12 | | 362 | | 11884 | | 391086 | | 12871192 |
| | | 6 | | 181 | | 5942 | | 195543 | |
| 80° | 11 | | 356 | | 11703 | | 385144 | | 12675649 |
| | | 17 | | 537 | | 17645 | | 580687 | |
| 70° | 10 | | 339 | | 11166 | | 367499 | | 12094962 |
| | | 27 | | 876 | | 28811 | | 948186 | |
| 60° | 9 | | 312 | | 10290 | | 338688 | | 11146776 |
| | | 36 | | 1188 | | 39101 | | 1286874 | |
| 50° | 8 | | 276 | | 9102 | | 299587 | | 9859902 |
| | | 44 | | 1464 | | 48203 | | 1586461 | |
| 40° | 7 | | 232 | | 7638 | | 251384 | | 8273441 |
| | | 51 | | 1696 | | 55841 | | 1837845 | |
| 30° | 6 | | 181 | | 5942 | | 195543 | | 6435596 |
| | | 57 | | 1877 | | 61783 | | 2033388 | |
| 20° | 4 | | 124 | | 4065 | | 133760 | | 4402208 |
| | | 61 | | 2001 | | 65848 | | 2167148 | |
| 10° | 2 | | 63 | | 2064 | | 67912 | | 2235060 |
| | | 63 | | 2064 | | 67912 | | 2235060 | |
| 0° | 0 | | 0 | | 0 | | 0 | | 0 |

Figure 1 Bürgi's "Exemplum" in *Fundamentum Astronomiæ* (fol. 36r, facsimile from `http://www.bibliotekacyfrowa.pl/dlibra`) and the same calculation in base 10 (from left to right and from top to bottom).

**Bürgi's *Artificium* or *Kunstweg*.** Bürgi's "skilful method" allows the simultaneous calculation of $n$ sine values ($s_0 = 0, s_1, \ldots, s_n$) on an equidistant grid of the interval $[0°, 90°]$

[1] Bürgi's manuscript can be consulted on the website of the Wrocław digital library: `http://www.bibliotekacyfrowa.pl/dlibra`. It has been carefully edited by Launert [23].

"dürch Theilüng eines rechten winckels in soüil theil als man will [partitioning the right angle in as many parts as one likes]" [23, p. 43]. Bürgi explains his method with calculations in base 60 for $n = 9$ in the "Exemplum" of Figure 1, to which we have added the same values in base 10 together with little arrows indicating the order in which the calculations are performed. The method is graphically displayed for $n = 4$ in Figure 2.




$$\begin{array}{l|l|l}
s_4 & \searrow & s'_4 = s'_3 + c_{\frac72} \\
 & c_{\frac72} = \frac{s_4}{2} & \\
s_3 & & s'_3 = s'_2 + c_{\frac52} \\
 & c_{\frac52} = c_{\frac72} + s_3 & \\
s_2 & & s'_2 = s'_1 + c_{\frac32} \\
 & c_{\frac32} = c_{\frac52} + s_2 & \\
s_1 & & s'_1 = c_{\frac12} \\
 & c_{\frac12} = c_{\frac32} + s_1 & \\
s_0 & & s'_0 = 0
\end{array}$$

Figure 2 Bürgi's algorithm ($n = 4$).

From arbitrary (but cleverly chosen) initial values

$$(s_0, s_1, \ldots, s_9) = (0,\ 2,\ 4,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12)\,, \tag{1}$$

for the sine values (to be scaled by $1/s_n$), Bürgi's algorithm computes iteratively

$$\begin{aligned} &\searrow c_{n-\frac12} = \tfrac12 s_n && \curvearrowleft s'_{k+1} = s'_k + c_{k+\frac12} \quad (k=0,\ldots,n-1) \\ &\curvearrowleft c_{k-\frac12} = c_{k+\frac12} + s_k \quad (k=n-1,\ldots,1) && s'_0 = 0 \end{aligned} \tag{2}$$

first by downward additions for approximations to the cosines $(c_{n-\frac12}, c_{n-\frac32}, \ldots, c_{\frac12})$ at the mid points, followed by upward additions for new approximations to the sines $(s'_0 = 0, s'_1, \ldots, s'_n)$ (again to be scaled by $1/s'_n$). From here, the method is iterated until a sufficiently high precision is reached. The starting value $c_{n-\frac12} = \frac{s_n}{2}$ in (2) is due to $c_{n+\frac12} = -c_{n-\frac12}$.

The method is apparently motivated by the relations (in modern notation)

$$\cos(x-\delta) - \cos(x+\delta) = 2\sin x \sin\delta\,, \quad \sin(x+\delta) - \sin(x-\delta) = 2\cos x \sin\delta\,, \tag{3}$$

which Bürgi called *Prosthaphaeresis* and proved in his Chapter 3 by two geometrical figures [23, p. 25]. In order to avoid tedious multiplications, Bürgi neglected the constant factor $K = 2\sin\delta$ throughout his calculations and normalized the final values $s_0^{(m)}$, $s_1^{(m)}$, $s_2^{(m)}, \ldots$ by one single division to $s_n^{(m)} = 1$.

The method seems to converge very well. For example, we see from the data of Figure 1 that the sequence of fractions

$$\frac{63}{362} = 0.1740,\ \frac{2064}{11884} = 0.17368,\ \frac{67912}{391086} = 0.1736498,\ \frac{2235060}{12871192} = 0.17364825,$$

converges rapidly to $\sin 10° = 0.17364817766693$. The maximal errors of all nine values are

| | Sinus 1 | Sinus 2 | Sinus 3 | Sinus 4 | Sinus 5 |
|---|---|---|---|---|---|
| err | 0.11602540 | 0.00414695 | 0.00015533 | 0.00000617 | 0.00000025 |
| ratio | | 27.978 | 26.698 | 25.195 | 24.423 |

with an approximate convergence rate of $1/25$.

**Proof of Convergence.** Since the method remained unknown through all these centuries, sound proofs of convergence were only given by contemporary authors (see, for example, [14], [30] and [25]) in the framework of modern theories. We give here a proof, as far as possible back in history, based on the early work of Joseph-Louis Lagrange (1736–1813) [21] and [20]. We reverse the formulas in (2) and obtain

$$c_{k+\frac{1}{2}} = s'_{k+1} - s'_k \quad \Rightarrow \quad s_k = c_{k-\frac{1}{2}} - c_{k+\frac{1}{2}} = -s'_{k+1} + 2s'_k - s'_{k-1}\,,$$

valid for $k=1,\ldots,n-1$. For $k=n$, we have $s_n = 2c_{n-\frac{1}{2}} = 2(s'_n - s'_{n-1})$. This determines a linear map $M\ :\ s = (s_1,\ldots,s_n) \mapsto s' = (s'_1,\ldots,s'_n)$. For example, in the case of Figure 2, the inverse of this map is given by

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -2 & 2 \end{pmatrix} \begin{pmatrix} s'_1 \\ s'_2 \\ s'_3 \\ s'_4 \end{pmatrix} \quad \text{or} \quad s = M^{-1}s'. \tag{4}$$

For discussing the question of convergence of $s \mapsto s' \mapsto s'' \mapsto s''' \ldots$ we search an eigenvector $v = (v_1,\ldots,v_n)$ of $M$, i.e., $Mv = \lambda v$ or $v = \lambda M^{-1} v$. For all rows of the matrix in (4), except the first and the last, this gives

$$v_k = -\lambda v_{k+1} + 2\lambda v_k - \lambda v_{k-1} \quad \text{or} \quad v_{k+1} + \Big(\frac{1}{\lambda} - 2\Big) v_k + v_{k-1} = 0\,. \tag{5}$$

This formula remains valid for *all* $k = 1,\ldots,n$, if we extend the vector $v$ with $v_0 = 0$ and $v_{n+1} = v_{n-1}$[2].

**How Lagrange found the eigenvectors of such linear maps?** The 23 year old Lagrange was the first to deal with a similar problem in his work on the theory of sound[3]. His method of discovery for finding an eigenvector can be explained as follows: equation (5) is a second-order linear difference equation and can be solved by the *ansatz* $v_k = r^k$ (another idea of Lagrange, see [20, p. 26]), which leads by linear superposition to

$$v_k = Ar_1^k + Br_2^k \quad \text{with} \quad r_1, r_2 \quad \text{roots of} \quad r^2 + \Big(\frac{1}{\lambda} - 2\Big) r + 1 = 0\,.$$

By Viète's formulae $r_1 r_2 = 1$, thus $r_2 = 1/r_1$. Since $v_0 = 0$, we get $B = -A$ and $v_k = A\cdot(r^k - \frac{1}{r^k})$. If $r$ is real, the condition $v_{n+1} = v_{n-1}$ cannot be satisfied. Therefore

[2] "(...) je considère que ces équations étant toutes semblables, on peut les exprimer généralement par (...) [I consider that these equations are all similar, we can express them generally by (...)]." [21, p. 74]

[3] [21, Sect. 1, Chap. III]. See also [16, pp. 28–29] for other boundary conditions and physical details.

$r$ must be on the complex unit circle, i.e., $r = e^{i\phi}$ for some real $\phi$ (which we choose between 0 and $\pi$). This leads, using Euler's famous equation[4], to

$$v_k = A \cdot (e^{ik\phi} - e^{-ik\phi}) = 2iA \cdot \sin k\phi.$$

The condition $v_{n+1} = v_{n-1}$ becomes, using (3),

$$\sin(n+1)\phi - \sin(n-1)\phi = 2\cos n\phi \sin\phi = 0\,.$$

Since $\sin\phi \neq 0$ (otherwise all $v_k = 0$) we have $\cos n\phi = 0$ so that $n\phi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, etc. This allows $n$ different values for $\phi = \frac{\pi}{2n}, \frac{3\pi}{2n}, \dots, \frac{(2n-1)\pi}{2n}$ and the eigenvectors become (see Fig. 3, left)

$$v_k^{(1)} = \sin\frac{k\pi}{2n},\quad v_k^{(2)} = \sin\frac{3k\pi}{2n},\quad v_k^{(3)} = \sin\frac{5k\pi}{2n},\quad \dots \quad (k = 1, \dots, n). \tag{6}$$

From $\frac{1}{\lambda} - 2 = -(r_1 + r_2)$ (again Viète's formulae) we get $\frac{1}{\lambda} = 2 - (e^{i\phi} + e^{-i\phi}) = 2 - 2\cos\phi = 4\sin^2\frac{\phi}{2}$ and thus the corresponding eigenvalues satisfy

$$\frac{1}{\lambda_1} = 4\,\sin^2\frac{\pi}{4n}\,,\qquad \frac{1}{\lambda_2} = 4\,\sin^2\frac{3\pi}{4n}\,,\qquad \frac{1}{\lambda_3} = 4\,\sin^2\frac{5\pi}{4n}\,,\qquad \dots\,. \tag{7}$$

The $n$ eigenvectors (6) span $\mathbb{R}^n$ and are orthogonal for $\langle x, y\rangle = \sum_{i=1}^{n-1} x_i y_i + \frac{1}{2}x_n y_n$.

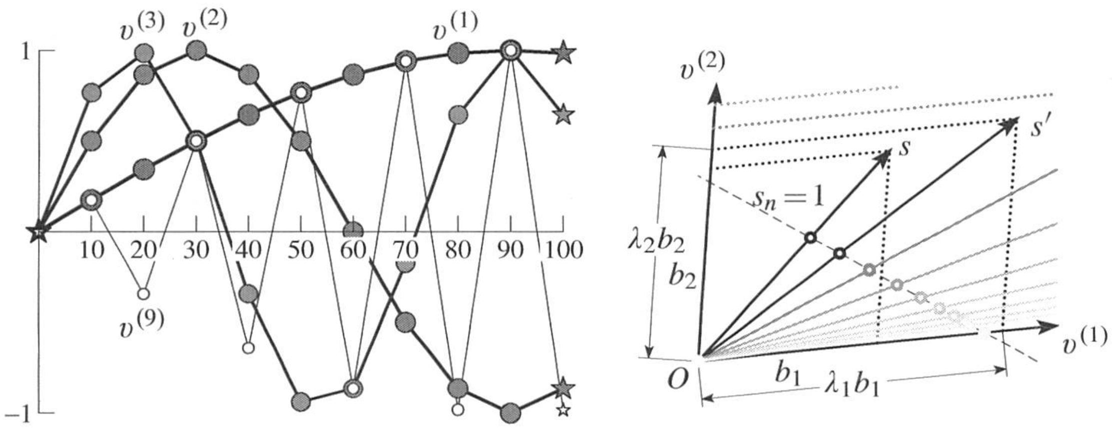



Figure 3 Eigenvectors $v^{(1)}, v^{(2)}, v^{(3)}$ and $v^{(9)}$ for $n = 9$ (left; the stars represent the extensions $v_0 = 0$ and $v_{n+1} = v_{n-1}$; convergence of the iteration $s, s', s'', s''', \dots$ with $\lambda_1 = 1.7$ and $\lambda_2 = 1.1$ (right).

**Conclusion.** We see that Bürgi's method is the same as the so-called "Power iteration" method for the eigenvector problem which is today standard knowledge. If $\lambda_1 > \lambda_2 > \cdots$

[4]"(...) comme nous l'enseignent les expressions exponentielles imaginaires des sinus & cosinus, si familières aujourd'hui aux Géomètres [as we learn from the imaginary exponential expressions of sine and cosine, so familiar today to the Geometers]" [21, p. 79].

and we write $s = b_1 v^{(1)} + b_2 v^{(2)} + \cdots$ (see Fig. 3, right), we have

$$s^{(m)} = b_1 \lambda_1^m v^{(1)} + b_2 \lambda_2^m v^{(2)} + \cdots = \lambda_1^m \Big( b_1 v^{(1)} + \Big(\frac{\lambda_2}{\lambda_1}\Big)^m b_2 v^{(2)} + \cdots \Big).$$

Thus we see that if $b_1 \neq 0$, we have convergence *in direction* to the eigenvector with the greatest eigenvalue $\lambda_1$. After normalization by $1/s_n^{(m)}$, the convergence is pointwise to the eigenvector $v^{(1)}$ (because $v_n^{(1)} = 1$), hence the coordinates of this normalized vector are $\sin \frac{k\pi}{2n}$ as claimed by Bürgi. If $b_2 \neq 0$, the convergence speed is $(\frac{\lambda_2}{\lambda_1})^m$.
For Bürgi's initial guess (1) we find the coefficients $b_i$ by numerical calculations:

$$10.970508,\ 0,\ 0.466576,\ -0.253301,\ 0,\ -0.124191,\ 0.101454,\ 0,\ 0.083971.$$

So we have by miracle $b_2 = 0$[5] and the convergence speed is $(\frac{\lambda_3}{\lambda_1})^m$, i.e., from (7):

$$\frac{\sin^2 \frac{\pi}{36}}{\sin^2 \frac{5\pi}{36}} = \frac{1}{23.5128}\ (n{=}9)\,, \qquad \frac{\sin^2 \frac{\pi}{360}}{\sin^2 \frac{5\pi}{360}} = \frac{1}{24.9848}\ (n{=}90)\,, \qquad \frac{1}{25}\ (n \to \infty). \tag{8}$$

This last limit would be $\frac{1}{9}$ if $b_2 \neq 0$.
We shall see in the next sections that Bürgi's method is closely related to a discovery of Johann Bernoulli (1667–1748).

## 2 Johann Bernoulli's successive involutes

> "Ce théorème remarquable est dû à Jean Bernouilli (...). [This remarkable theorem is due to Johann Bernoulli.]"
>
> (Siméon Denis Poisson, [27, p. 440])

One hundred and ten years after Bürgi's death appeared the *Opera Omnia* [6] of Johann Bernoulli in four volumes published by Marc-Michel Bousquet (1696–1762). In volume IV ("Quo continentur ANEKΔOTA") are collected unpublished manuscripts which Johann judged interesting for posterity. The article CLXV describes a surprising method "successiva et alternante" for "cyclometricum" (calculation of $\pi$) based on a fixed-point property of the cycloid[6].

**Bernoulli's Theorem.** Let $ADB$ (see Fig. 4) be an arbitrary curve ("curva quælibet") whose tangents in $A$ and $B$ are perpendicular. Produce the tangent $BF$ and the parallel axis $CA$ to infinity. Then describe through $A$ the involute $AEF$ of the curve $ADB$ ending in $F$, describe through $F$ the involute $FGH$ of the curve $AEF$ ending in $H$ and continue alternately to infinity. Bernoulli claims (without a direct proof) that at infinity we so obtain

[5] This phenomenon occurs for all $n \equiv 0 \pmod 3$.

[6] De evolutione successiva et alternante curva cujuscunque in infinitum continuata, tandem Cycloidem generante; schediasma cyclometricum [On the successive and alternating development of any curve continued to infinity eventually generating a cycloid; cyclometric scheme] [6, IV, pp. 98–108]. A handwritten version of this text (without title) is conserved in the University Library of Basel [7].

identical cycloids ("Cycloides identicas"), whatever was the initial curve[7]. According to Joseph E. Hofmann (1900–1973) "es ist das erste Beispiel, das erkennen lässt, dass sich eine Folge wohldefinierter Kurvenbögen einem Grenzbogen nähert[8]".

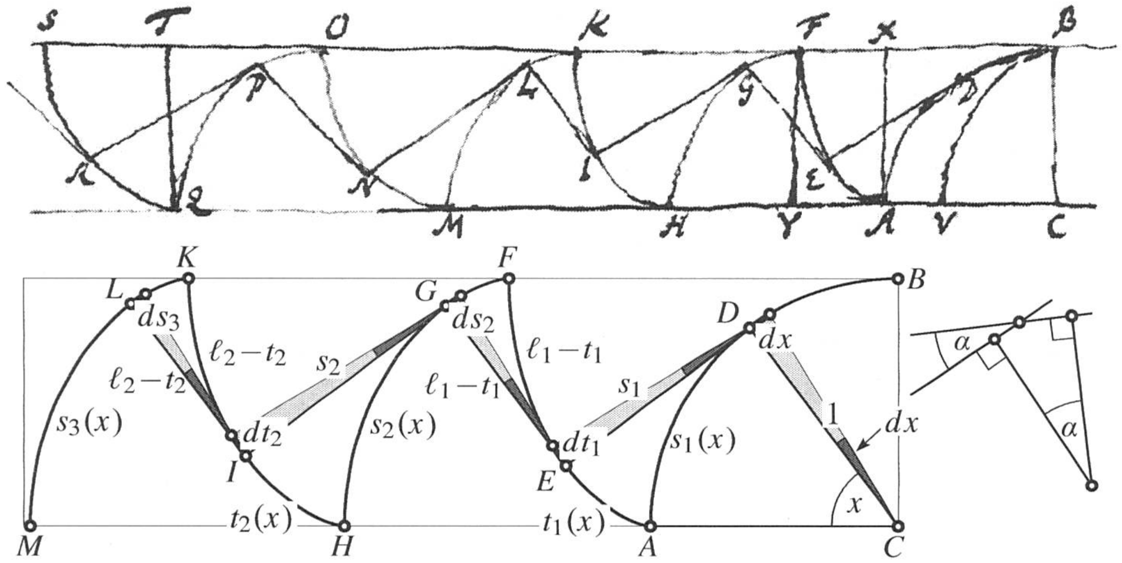



Figure 4 Illustration from [7, fol. 245] (above), and a modern drawing (below).

**Computations for the case of a circle.** Johann computes the arc lengths for the case when the curve $ADB$ is the quarter of a circle with radius 1. We use the abbreviations[9]

$$\text{arc}\, AB = \frac{\pi}{2} = a, \quad \text{arc}\, AF = \ell_1, \quad \text{arc}\, HK = \ell_2, \quad \text{arc}\, MO = \ell_3, \quad \text{etc.}$$

We choose a point $D$ on $AB$ and denote by $x$ the angle between the normal to the curve at $D$ and the axis $CA$. The successive tangents to these curves lead to a polygon $D$, $E$, $G$, $I$, $L$, $N$, $P$, $R$, etc. We denote the arc lengths $AD$, $AE$, $HG$, $HI$, $ML$, ..., which are all zero for $x = 0$, by $s_1(x)$, $t_1(x)$, $s_2(x)$, $t_2(x)$, $s_3(x)$, ... respectively[10]. By the property of the involutes, the tangents $DE$, $EG$, $GI$, $IL$, ... are equal to $s_1$, $\ell_1 - t_1$, $s_2$, $\ell_2 - t_2$, etc. By the property of orthogonal angles, all dark angles are mutually equal and equal to $dx$, hence we obtain

$$dt_1 = s_1\, dx, \quad ds_2 = (\ell_1 - t_1)\, dx, \quad dt_2 = s_2\, dx, \quad ds_3 = (\ell_2 - t_2)\, dx, \quad \dots .$$

Beginning with $s_1(x) = x$ (the arc length of the circle), we integrate these equations one after the other as

$$t_j(x) = \int_0^x s_j(\xi)\, d\xi\,, \qquad s_{j+1}(x) = \int_0^x (\ell_j - t_j(\xi))\, d\xi \tag{9}$$

[7] We shall see that the cycloids are identical for *all* initial curves, if we normalize the distance $CB$ to 1. Also, $s_1(x)$ is only well defined, if the slope of the curve decreases monotonically.

[8] [18, p. 98].

[9] The notation $a$ is that of Joh. Bernoulli, Euler [11] and Legendre [24]. For $\ell_1$, $\ell_2$, $\ell_3$, ... Bernoulli wrote $b$, $c$, $e$, ... (see Fig. 5) while Euler, Lagrange and Legendre used $b$, $b'$, $b''$, ....

[10] Euler [11] wrote for these functions $s$, $t$, $s'$, $t'$, $s''$, ....

and obtain (for a reproduction of Johann's formulas, see Fig. 5)

$$
\begin{array}{lcl}
t_1(x) = \frac{x^2}{2!} & \longleftarrow & s_1(x) = x \\
 & \rightleftarrows & s_2(x) = \ell_1 x - \frac{x^3}{3!} \\
t_2(x) = \ell_1 \frac{x^2}{2!} - \frac{x^4}{4!} & & \\
 & \rightleftarrows & s_3(x) = \ell_2 x - \ell_1 \frac{x^3}{3!} + \frac{x^5}{5!} \\
t_3(x) = \ell_2 \frac{x^2}{2!} - \ell_1 \frac{x^4}{4!} + \frac{x^6}{6!} & & \\
 & \longrightarrow & s_4(x) = \ell_3 x - \ell_2 \frac{x^3}{3!} + \ell_1 \frac{x^5}{5!} - \frac{x^7}{7!}.
\end{array}
\tag{10}
$$

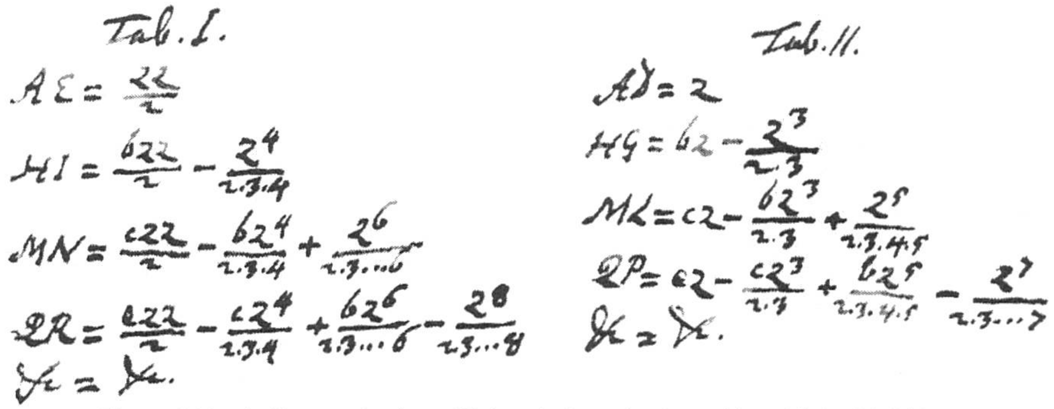



Figure 5 Facsimile reproduction of Johann's formulas for $s_j(z)$, $t_j(z)$ [7, fol. 245v].

The conditions $t_j(a) = \ell_j$ or $0 = \ell_j - t_j(a)$ allow the lengths $\ell_j$ to be computed recursively as

$$
\begin{array}{ll}
0 = \ell_1 - \frac{a^2}{2!} & \Rightarrow \ell_1 = \frac{a^2}{2!} \\
0 = \ell_2 - \ell_1 \frac{a^2}{2!} + \frac{a^4}{4!} & \Rightarrow \ell_2 = \frac{5a^4}{4!} \\
0 = \ell_3 - \ell_2 \frac{a^2}{2!} + \ell_1 \frac{a^4}{4!} - \frac{a^6}{6!} & \Rightarrow \ell_3 = \frac{61a^6}{6!} \quad \text{etc.}
\end{array}
\tag{11}
$$

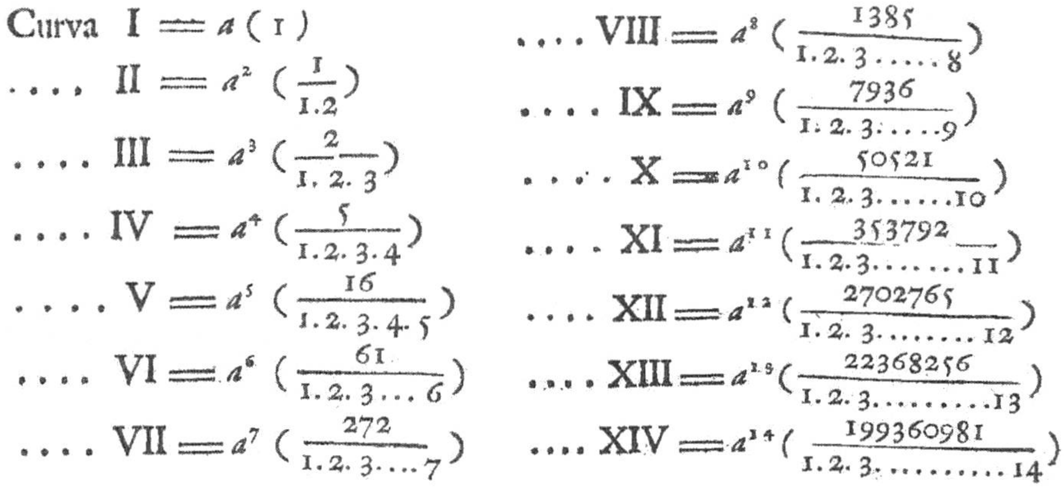
Finally, after calculating $s_j(a)$ from (10), we get the list of all the arc lengths:

$$
AF = \frac{a^2}{2!},\ FH = \frac{2a^3}{3!},\ HK = \frac{5a^4}{4!},\ KM = \frac{16a^5}{5!},\ MO = \frac{61a^6}{6!},\ \text{etc.} \tag{12}
$$

(see Johann's longer list in Fig. 6 and the still longer list in (26) below).

**Rectifying the circle.** Equating, for example, the curve lengths VII and VIII, and "dividendo per $a^7$", we obtain $a = \frac{8\cdot 272}{1385}$ and hence $2a = \pi \simeq \frac{4352}{1385} = 3.14224$, so that "nostra analogia tantillo minor est, quam *Archimedea*" [6, IV, p. 103]. The best approximations are obtained by comparing the curves XII–XIII and XIII–XIV with the result

$$
\frac{70271890}{22368256} = 3.14159003 < \pi < \frac{626311168}{199360981} = 3.14159353. \tag{13}
$$



Curva I $= a\,(1)$

.... II $= a^2\,(\frac{1}{1.2})$

.... III $= a^3\,(\frac{2}{1.2.3})$

.... IV $= a^4\,(\frac{5}{1.2.3.4})$

.... V $= a^5\,(\frac{16}{1.2.3.4.5})$

.... VI $= a^6\,(\frac{61}{1.2.3\ldots6})$

.... VII $= a^7\,(\frac{272}{1.2.3\ldots7})$

.... VIII $= a^8\,(\frac{1385}{1.2.3\ldots\ldots8})$

.... IX $= a^9\,(\frac{7936}{1.2.3\ldots\ldots9})$

.... X $= a^{10}\,(\frac{50521}{1.2.3\ldots\ldots10})$

.... XI $= a^{11}\,(\frac{353792}{1.2.3\ldots\ldots11})$

.... XII $= a^{12}\,(\frac{2702765}{1.2.3\ldots\ldots12})$

.... XIII $= a^{13}\,(\frac{22368256}{1.2.3\ldots\ldots13})$

.... XIV $= a^{14}\,(\frac{199360981}{1.2.3\ldots\ldots14})$

Figure 6 Table from [6, IV, p. 106]: all values are correct.

*Remark.* We can also compare the values contained in Figure 6 with the known length of the limiting cycloidal arc, which is $\frac{4}{\pi}$ (see (24)). So, by extracting a high-order root we get for "Curva XIV" the better value

$$\pi \sim 2\sqrt[15]{\frac{2\cdot 14!}{199360981}} = 3.14159266818 \quad \text{with an error of} \quad 1.46\cdot 10^{-8}. \tag{14}$$

But obviously Johann preferred to calculate $\pi$ "sine extractione radicum, & sine comparatione Cycloidis" [6, IV, p. 102].

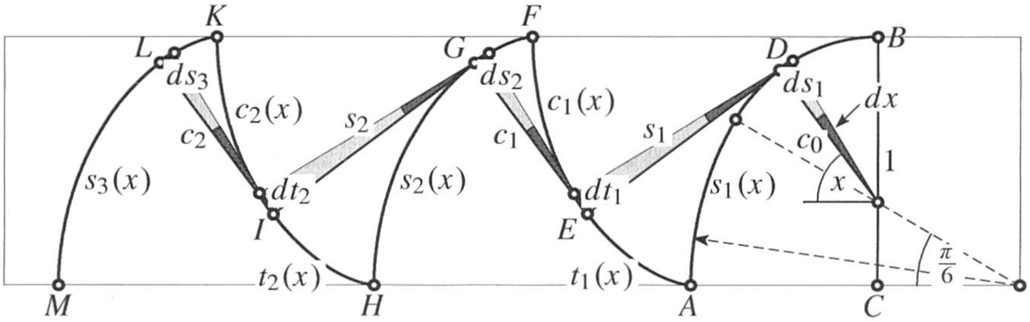



Figure 7 The initial curve corresponding to Bürgi's values (1) and involutes.

**Relation with Bürgi's algorithm.** If we introduce for the superior part of the curve lengths in Figure 4

$$c_j(x) = \ell_j - t_j(x) = \int_0^a s_j(\xi)\,d\xi - \int_0^x s_j(\xi)\,d\xi = \int_x^a s_j(\xi)\,d\xi\,,$$

then Bernoulli's algorithm (9) becomes nicely symmetric

$$c_j(x) = \circlearrowleft\!\!\int_x^a s_j(\xi)\,d\xi\,, \qquad s_{j+1}(x) = \circlearrowleft\!\!\int_0^x c_j(\xi)\,d\xi \tag{15}$$

and turns out to be identical with Bürgi's algorithm (2) in the case when $n\to\infty$ and when the constant $K = 2\sin\delta$, which Bürgi had at first neglected, becomes the $d\xi$. We further introduce $c_0(x)$ for the derivative of $s_1(x)$, which is also the corresponding radius of curvature. Bürgi's clever initial choice (1), where the values $s_i$ increase twice as fast below 30° than above, would correspond to

$$c_0(x) = \frac{2}{3} \cdot \begin{cases} 2 & \text{if } 0 \le x < \frac{a}{3} \\ 1 & \text{if } \frac{a}{3} \le x \le a\,, \end{cases} \qquad s_1(x) = \frac{2}{3} \cdot \begin{cases} 2x & \text{if } 0 \le x < \frac{a}{3} \\ x + \frac{a}{3} & \text{if } \frac{a}{3} \le x \le a\,. \end{cases} \tag{16}$$

The factor $\frac{2}{3}$ assures the condition $CB = 1$. The corresponding initial curve then consists of two circles of radii $\frac{4}{3}$ and $\frac{2}{3}$ respectively. Rapid convergence towards "Cycloides identicas" can be observed in Figure 7.

## 3 Convergence proof

"(...) il est, en général, plus long et plus pénible de résoudre complètement une question relative à un nombre indéterminé de points matériels, que la même question dans laquelle on suppose immédiatement ce nombre infini [It is, in general, longer and more painful to solve completely a question related to a finite number of material points, than the same question in which this number is immediately supposed infinite.]"

(Siméon Denis Poisson, *Théorie mathématique de la chaleur*, 1835, p. 171)

The first proof for Bernoulli's affirmation is due to Leonhard Euler (1707–1783) in [11]. Lagrange, in an unpublished manuscript from 1780, asserts that Euler's method "doesn't bring and cannot bring in mind all the light and all the conviction that one can desire on this subject" and writes his own proof. Later proofs were given by Legendre [24, pp. 541–544] and Poisson [27, pp. 431–440]. But it was not until 1844 that appeared the first short proof by Puiseux [28, pp. 397–399][11].

All these proofs show, by various methods, that the arc lengths converge to

$$s_j(x) \to \text{Const} \cdot \sin x \quad \text{(and also} \quad c_j(x) \to \text{Const} \cdot \cos x) \quad \text{for } j \to \infty. \tag{17}$$

With this information, say, for the curves $MLK$ in Figures 4 and 7, we know at every point $L$

- the arc length $ML$, as well as
- the direction (defined by the angle $x$).

Therefore the curves are uniquely determined. We know that the involute of a cycloid is also a cycloid[12] and that the involute of a hypocycloid/epicycloid is also a similar curve. This shows that the curves in Figure 8 (because of Eucl. III.20) satisfy

$$\text{arc}\, LM = LN = LP + PN = \begin{cases} 4r \cdot \sin x & \text{(cycloid, left)}, \\ 2(r_1 + r_2)\sin x & \text{(hypocycloid, right)}, \end{cases} \tag{18}$$

and so have both of these properties. Thus the equations (17) lead to Johann's Theorem and to Euler's generalization of this result to hypo/epicycloid.

[11] Lagrange's manuscript together with an introduction will be published elsewhere (see [22]).

[12] This result is due to Huygens in his *Horlogium oscillatorium*, see also [17, Thms. 1 and 3].

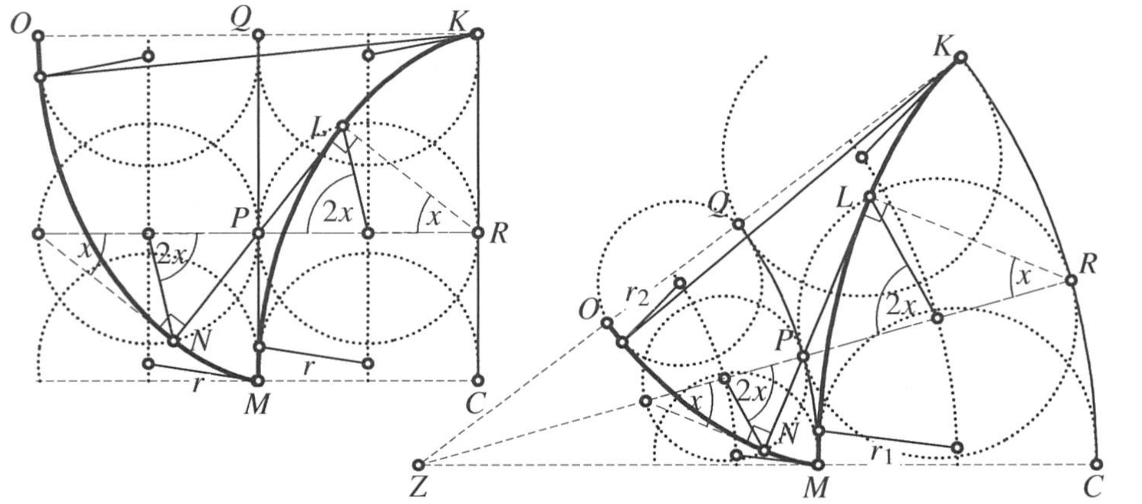



Figure 8 Cycloid (left) and hypocycloid (right).

**Proof of (17).** Since Johann's algorithm is the same as Bürgi's (for $n \to \infty$) and since (17) is the same as Bürgi's claim in Section 1, the proof of (17) is immediate from what we have proved there. Following the advice of Poisson (see quotation), the continuous case is even simpler to understand.

Inspired by the eigenvectors (6), letting $\frac{k\pi}{2n} \to x$ (for $n \to \infty$, $0 \le x \le \frac{\pi}{2}$), we consider the basis functions

$$\sin x\,,\ \sin 3x\,,\ \sin 5x\,,\ \ldots \quad \text{(and also} \quad \cos x\,,\ \cos 3x\,,\ \cos 5x\,,\ \ldots)\,.$$

Linear combinations of these functions are the *Fourier series on the interval* $[0, \frac{\pi}{2}]$. Because of the integration formulas

$$\circlearrowleft\!\int_x^{\frac{\pi}{2}} \sin k\xi\, d\xi = -\frac{1}{k}\cos k\xi\Big|_x^{\frac{\pi}{2}} = \frac{1}{k}\cos kx \quad (\text{for } k = 1, 3, 5, \ldots) \tag{19}$$

(observe that $\cos k\frac{\pi}{2} = 0$ for odd $k$) and

$$\circlearrowright\!\int_0^{x} \cos k\xi\, d\xi = \frac{1}{k}\sin kx \quad (\text{for all } k), \tag{20}$$

we see that *the downward integration* $(\circlearrowleft)$ *transforms any series in* $\sin kx$ *into a series in* $\cos kx$ *by dividing the coefficients by* $k$*, and the upward integration* $(\circlearrowright)$ *transforms any series in* $\cos kx$ *into a series in* $\sin kx$*, again by dividing the coefficients by* $k$*.*

In view of the integrations in (15), we develop $c_0(x)$ into a series in $\cos kx$ and $s_1(x)$ into a series in $\sin kx$ :

$$c_0(x) = a_1\cos x + a_3\cos 3x + a_5\cos 5x + \cdots \quad \text{with} \quad a_k = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} c_0(x)\cos kx\, dx \tag{21}$$

$$s_1(x) = b_1\sin x + b_3\sin 3x + b_5\sin 5x + \cdots \quad \text{with} \quad b_k = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} s_1(x)\sin kx\, dx\,. \tag{22}$$

The integral formulas for the coefficients, together with their standard proof using orthogonality, were independently discovered by Euler [13] and Fourier (*Théorie analytique de la chaleur*, 1822).

**Formulas for Bernoulli's case.** For the functions (10), which Johann Bernoulli had computed, we have $s_1(x) = x$ and hence start with $c_0(x) = 1$. The coefficients $a_k$ in (21) become $1\,,\ -\frac{1}{3}\,,\ \frac{1}{5}\,,\ldots$ and we obtain, by the continued integrations of (15) and alternative use of (20) and (19),

$$\begin{aligned} c_0(x) &= \tfrac{4}{\pi}\big(+\cos x - \tfrac{1}{3}\cos 3x + \tfrac{1}{5}\cos 5x - \tfrac{1}{7}\cos 7x + \cdots\big) \\ s_1(x) &= \tfrac{4}{\pi}\big(+\sin x - \tfrac{1}{3^2}\sin 3x + \tfrac{1}{5^2}\sin 5x - \tfrac{1}{7^2}\sin 7x + \cdots\big) \\ c_1(x) &= \tfrac{4}{\pi}\big(+\cos x - \tfrac{1}{3^3}\cos 3x + \tfrac{1}{5^3}\cos 5x - \tfrac{1}{7^3}\cos 7x + \cdots\big) \\ s_2(x) &= \tfrac{4}{\pi}\big(+\sin x - \tfrac{1}{3^4}\sin 3x + \tfrac{1}{5^4}\sin 5x - \tfrac{1}{7^4}\sin 7x + \cdots\big) \\ c_2(x) &= \tfrac{4}{\pi}\big(+\cos x - \tfrac{1}{3^5}\cos 3x + \tfrac{1}{5^5}\cos 5x - \tfrac{1}{7^5}\cos 7x + \cdots\big) \end{aligned} \tag{23}$$

and so on. Therefore, for $j \to \infty$, we have in the limit

$$s_1(x), s_2(x), s_3(x), \ldots \to \frac{4}{\pi}\sin x \quad \text{and} \quad c_0(x), c_1(x), c_2(x), \ldots \to \frac{4}{\pi}\cos x\,, \tag{24}$$

since all coefficients except the first one tend to zero. The coefficients of $\sin 3x$ converge most slowly, therefore we have convergence ratio $\frac{1}{3^2} = \frac{1}{9}$.
The convergence of the Fourier series for $c_0(x)$, $s_1(x)$ and $c_1(x)$ is shown in Figure 9 (left, where also a nice Gibbs phenomenon can be seen).

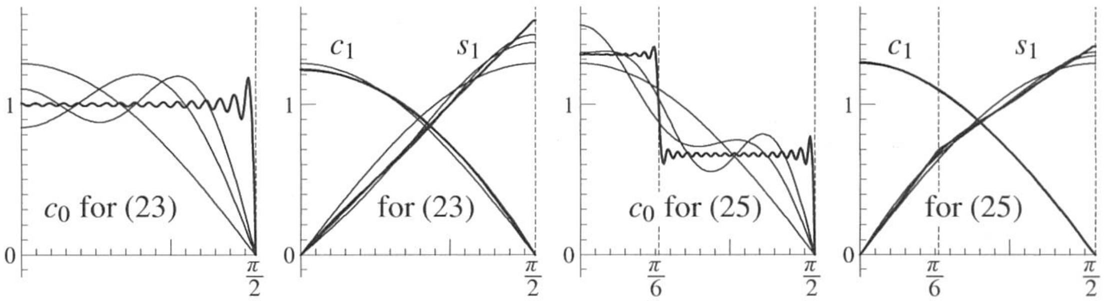



Figure 9 Convergence of the Fourier series for Bernoulli's (left) and Bürgi's (right) case: $c_0$, $s_1$ and $c_1$ (1,2,3 and 33 non-zero terms).

**Formulas for Bürgi's case.** If we compute the $a_k$ for the initial functions (16), we have the nice surprise that $a_3 = 0$. The full series becomes

$$\begin{aligned} c_0(x) &= \tfrac{4}{\pi}\big(+\cos x + \tfrac{1}{5}\cos 5x - \tfrac{1}{7}\cos 7x - \tfrac{1}{11}\cos 11x + \tfrac{1}{13}\cos 13x + \cdots\big) \\ s_1(x) &= \tfrac{4}{\pi}\big(+\sin x + \tfrac{1}{5^2}\sin 5x - \tfrac{1}{7^2}\sin 7x - \tfrac{1}{11^2}\sin 11x + \tfrac{1}{13^2}\sin 13x + \cdots\big) \\ c_1(x) &= \tfrac{4}{\pi}\big(+\cos x + \tfrac{1}{5^3}\cos 5x - \tfrac{1}{7^3}\cos 7x - \tfrac{1}{11^3}\cos 11x + \tfrac{1}{13^3}\cos 13x + \cdots\big) \end{aligned} \tag{25}$$

and so on (see Fig. 9, right). The dominant error in $s_1(x)$ is due to the term $\sin 5x$, so we have faster convergence with ratio $\frac{1}{5^2} = \frac{1}{25}$ as observed in (8).
Nicollier found further initial conditions with still faster convergence in [25].

# 4 Connection with the Euler numbers

"Haecque consideratio ad plurimas alias speculationes non spernandas perducere poterit. [This consideration [the Euler numbers] can lead to many other speculations that should not be despised.]"

(Leonhard Euler, *Summarium* of [11])

**Euler zigzag numbers.** We recall (and extend a little bit) the sequence of numbers which appear in Johann's list of Figure 6:

$$\begin{aligned} &E_0 = 1\,,\ E_1 = 1\,,\ E_2 = 1\,,\ E_3 = 2\,,\ E_4 = 5\,,\ E_5 = 16\,,\ E_6 = 61\,,\ E_7 = 272\,,\\ &E_8 = 1385\,,\ E_9 = 7936\,,\ E_{10} = 50521\,,\ E_{11} = 353792\,,\ E_{12} = 2702765\,,\\ &E_{13} = 22368256\,,\ E_{14} = 199360981\,,\ E_{15} = 1903757312\,,\ E_{16} = 19391512145\,,\\ &E_{17} = 209865342976\,,\ E_{18} = 2404879675441\,,\ E_{19} = 29088885112832\,,\\ &E_{20} = 370371188237525\,,\ E_{21} = 4951498053124096\,,\ \ldots\,. \end{aligned} \tag{26}$$

These numbers have been rediscovered several times and are often called the *Euler zigzag*[13] *numbers* by authors who were not aware of Johann's contribution.

**Euler's Summation Formulas.** We set in (23) $x = 0$ for $c_0, c_1, c_2, \ldots$ and $x = a$ for $s_1, s_2, s_3, \ldots$ and get from (12) the remarkable formulas

$$\begin{array}{l|l} 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\mathbf{1}\pi}{0!\cdot 2^2} & 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{\mathbf{1}\pi^2}{1!\cdot 2^3} \\ 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \cdots = \frac{\mathbf{1}\pi^3}{2!\cdot 2^4} & 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \cdots = \frac{\mathbf{2}\pi^4}{3!\cdot 2^5} \\ 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} - \cdots = \frac{\mathbf{5}\pi^5}{4!\cdot 2^6} & 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \cdots = \frac{\mathbf{16}\pi^6}{5!\cdot 2^7} \\ 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} + \frac{1}{9^7} - \cdots = \frac{\mathbf{61}\pi^7}{6!\cdot 2^8} & 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \cdots = \frac{\mathbf{272}\pi^8}{7!\cdot 2^9} \end{array}$$

or, in general,

$$1 - \frac{1}{3^{2\ell+1}} + \frac{1}{5^{2\ell+1}} - \frac{1}{7^{2\ell+1}} + \frac{1}{9^{2\ell+1}} - \cdots = \frac{\pi^{2\ell+1}}{(2\ell)!\cdot 2^{2\ell+2}}\, E_{2\ell}\,, \tag{27}$$

$$1 + \frac{1}{3^{2\ell}} + \frac{1}{5^{2\ell}} + \frac{1}{7^{2\ell}} + \frac{1}{9^{2\ell}} + \cdots = \frac{\pi^{2\ell}}{(2\ell-1)!\cdot 2^{2\ell+1}}\, E_{2\ell-1}\,. \tag{28}$$

**Euler numbers.** Formula (27) was for Euler [9] the original motivation for studying the numbers (26) with *even* indices, called the *Euler numbers*[14] (see Fig. 10). Formulas (11), together with $\ell_k = \frac{E_{2k}a^{2k}}{(2k)!}$ (see (12)), show that they obey recursions like

$$0 = \binom{6}{6}E_6 - \binom{6}{4}E_4 + \binom{6}{2}E_2 - \binom{6}{0}E_0\,. \tag{29}$$

[13] This terminology has been popularized by John H. Conway and Richard K. Guy during the 90s [26].

[14] It seems that this terminology was coined by James Joseph Sylvester (1814–1897): "Following the accepted *Continental* notation, I denote by $B_n$ (...)" and "I call the numbers $E_1$, $E_2$, ..., $E_n$ Euler's 1st, 2nd, ... $n$th numbers, as Euler was apparently the first to bring them into notice" (*Mathematical Papers* II, p. 254 & p. 262).

| | | | |
|---|---|---|---|
| α = 1 | | ζ = 50521 | |
| β = 1 | | η = 2702765 | |
| γ = 5 | | θ = 199360981 | |
| δ = 61 | | ι = 19391512145 | |
| ε = 1385 | | κ = 2404879661671 | &c. |

Figure 10 The Euler numbers printed in Euler's *Institutiones calculi differentialis* (1755) [10, part. 2, §224]. Only $\kappa = E_{18}$ is slightly wrong.

**Bernoulli numbers.** The numbers (26) with *odd* indices are connected to Euler's great discovery

$$1 + \frac{1}{2^{2\ell}} + \frac{1}{3^{2\ell}} + \frac{1}{4^{2\ell}} + \frac{1}{5^{2\ell}} + \frac{1}{6^{2\ell}} + \cdots = (-1)^{\ell-1} \frac{(2\pi)^{2\ell}}{2(2\ell)!} B_{2\ell}\,, \tag{30}$$

where $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, ... are the *Bernoulli numbers*[15]. Indeed, if we multiply (30) by $\frac{1}{2^{2\ell}}$ and substract from itself, we obtain (28) together with

$$E_{2\ell-1} = (-1)^{\ell-1} \cdot \frac{(2^{2\ell}-1)2^{2\ell}}{2\ell} \cdot B_{2\ell}\,. \tag{31}$$

The right-hand formulas of (10) show that they satisfy recursions like

$$E_5 = \binom{5}{4} E_4 - \binom{5}{2} E_2 + \binom{5}{0} E_0\,. \tag{32}$$

**Asymptotic values.** Johann's approximation method (14) can now be written

$$\pi \sim 2 \sqrt[n+1]{\frac{2 \cdot n!}{E_n}} \qquad \text{or} \qquad E_n \sim 2\Big(\frac{2}{\pi}\Big)^{n+1} n! \tag{33}$$

with a relative error of size $3^{-n}$. The estimates of (13) generalize to

$$\frac{2 \cdot (2n+1) \cdot E_{2n}}{E_{2n+1}} < \pi < \frac{2 \cdot (2n+2) \cdot E_{2n+1}}{E_{2n+2}}\,.$$

**An arbitrary angle *b*.** In the second part of his work [11, §28–45], Euler generalized Johann's Theorem to the case where the angle $a = \frac{\pi}{2}$ is replaced by a value $b$ other than $\frac{\pi}{2}$. An illustration is given in Figure 11 for the case where $b < a$ and the initial curve is a circle of radius 1. Bernoulli's algorithm (15) becomes here

$$\tilde{c}_j(y) = \circlearrowleft\!\!\int_y^b \tilde{s}_j(\eta)\, d\eta\,, \qquad \tilde{s}_{j+1}(y) = \circlearrowright\!\!\int_0^y \tilde{c}_j(\eta)\, d\eta\,.$$

[15]They have been named by Euler: "isti numeri, qui ab Inventore *Iacobo Bernoullio* vocari solent Bernoulliani" [10, part. 2, §122].

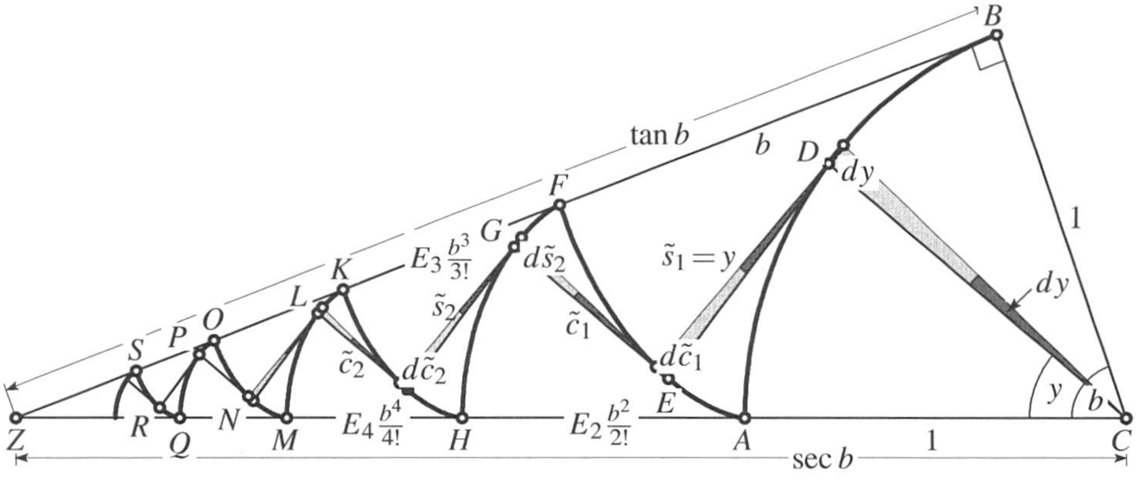


Figure 11 Bernoulli's algorithm for $b < a = \frac{\pi}{2}$.

For the computation of the new arc lengths $\tilde{c}_j$ and $\tilde{s}_{j+1}$, we use the coordinate transforms $y = \frac{b}{a} \cdot x$, $\eta = \frac{b}{a} \cdot \xi$, $d\eta = \frac{b}{a} \cdot d\xi$ in the integrals and obtain recursively, by starting from $\tilde{s}_1(y) = y = \frac{b}{a} \cdot x = \left(\frac{b}{a}\right)^1 s_1(x)$, the formulas

$$\tilde{c}_j(y) = \left(\frac{b}{a}\right)^{2j} c_j(x), \qquad \tilde{s}_{j+1}(y) = \left(\frac{b}{a}\right)^{2j+1} s_{j+1}(x). \tag{34}$$

The arc lengths $s_j$ and $c_j$, due to the ratio $d\eta/d\xi = \frac{b}{a}$, receive an additional factor $\frac{b}{a}$ at each integration. They thus tend to zero for $b < a$, to infinity for $b > a$. But when the involutes are continuously rescaled by $\frac{a^2}{b^2}$, they tend to finite curves. From (24) we get

$$\lim_{j\to\infty} \left(\frac{a}{b}\right)^{2j} \tilde{s}_j(y) = \frac{a}{b}\,\frac{4}{\pi}\,\sin\frac{a}{b}\,y \quad \text{and} \quad \lim_{j\to\infty} \left(\frac{a}{b}\right)^{2j} \tilde{c}_j(y) = \frac{4}{\pi}\,\cos\frac{a}{b}\,y\,.$$

So the limit is a hypocycloidal arc if $b < a$ (where in equation (18) $r_1 = \frac{a}{b} r_2$, compare with the right picture of Fig. 8) and an epicycloid for $b > a$ (Euler [11, §45]: "quae sunt proprietates epicycloidum et hypocycloidum").

**The tangent and secant functions.** Another motivation for Euler to introduce his numbers was the study of the secant function[16]. Figure 11 presents an elegant access to its series. We have $b < a = \frac{\pi}{2}$ so that the iterated involutes zigzag inside a right-angled triangle with hypotenuse $\sec b = \frac{1}{\cos b}$ and legs 1, $\tan b$. With (34) the arc lengths (12) become

$$BA = BF = b\,, \; AF = AH = \left(\frac{b}{a}\right)^2 \frac{a^2}{2!} = \frac{b^2}{2!}\,, \; FH = FK = \left(\frac{b}{a}\right)^3 \frac{2a^3}{3!} = \frac{2b^3}{3!}\,,$$

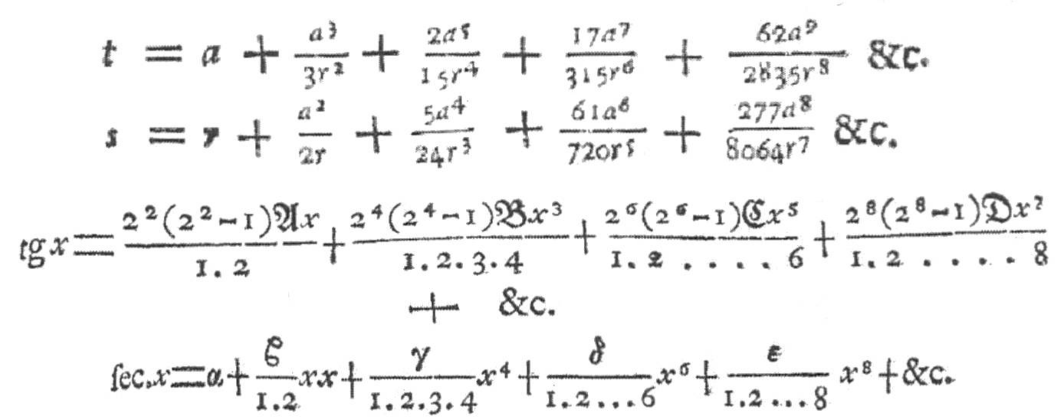
etc., the same expressions as before with $a$ replaced by $b$. From this picture we can immediately read off the infinite series for $\tan b$ and $\sec b$. Since $b$ is arbitrary, we replace this

[16] "Per hos autem numeros Bernoullianos secans exprimi non potest, sed requirit alios numeros, qui in summas potestatum reciprocarum imparium ingrediuntur" ["The secant can not be expressed through Bernoulli numbers but needs other numbers which appear in the sums of reciprocal powers of odd numbers."] [10, part. 2, §224].

letter by the more common $x$ and have

$$\sec x = \frac{1}{\cos x} = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k}, \qquad \tan x = \sum_{k=0}^{\infty} \frac{E_{2k+1}}{(2k+1)!} x^{2k+1}. \tag{35}$$

James Gregory (1638–1675) knew these results but divided out common factors so that no structure is visible and the series "maxime videntur irregulares". On the contrary, Euler reduced the series "ad facilem progressionis" (see Fig. 12).



$$t = a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8} \; \&c.$$
$$s = r + \frac{a^2}{2r} + \frac{5a^4}{24r^3} + \frac{61a^6}{720r^5} + \frac{277a^8}{8064r^7} \; \&c.$$
$$tg\,x = \frac{2^2(2^2-1)\mathfrak{A}x}{1.2} + \frac{2^4(2^4-1)\mathfrak{B}x^3}{1.2.3.4} + \frac{2^6(2^6-1)\mathfrak{C}x^5}{1.2\ldots\ldots6} + \frac{2^8(2^8-1)\mathfrak{D}x^7}{1.2\ldots\ldots8} + \&c.$$
$$sec.x = \alpha + \frac{\beta}{1.2}xx + \frac{\gamma}{1.2.3.4}x^4 + \frac{\delta}{1.2\ldots6}x^6 + \frac{\varepsilon}{1.2\ldots8}x^8 + \&c.$$

Figure 12 The tangent and secant series by James Gregory (letter to D. Collins, 15 Februarii 1671) and by Euler ([10, part. 2, §221, §224]).

**Exponential generating function of the Euler zigzag numbers.** There exists a close connection between the developments of $\tan x$ and $\sec x$ seen for the first time, according to Désiré André (1840–1917), by Eugène Catalan (1814–1894) [2, §12]. Indeed, by adding up both series (35) we obtain the exponential generating function (e. g. f.) of the Euler zigzag numbers. Because of

$$\frac{1}{\cos x} + \tan x = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right), \qquad \text{we have} \qquad \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) = \sum_{k=0}^{\infty} \frac{E_k}{k!} x^k. \tag{36}$$

For a geometric proof of the first identity, see Figure 13.

**Recursive calculation of the $E_k$.** Differentiating $y(x) = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$ we obtain

$$y' = \frac{1}{2}\left(1 + \tan^2\left(\tfrac{x}{2} + \tfrac{\pi}{4}\right)\right) \qquad \text{or} \qquad 2y' = 1 + y^2, \quad y(0) = \tan\tfrac{\pi}{4} = 1. \tag{37}$$

This is an initial value problem for a differential equation, to which we apply an idea of Euler [12, §663]. We develop

$$\begin{aligned}
y &= E_0 + \frac{E_1}{1!}x + \frac{E_2}{2!}x^2 + \cdots \\
2y' &= 2E_1 + 2\frac{E_2}{1!}x + 2\frac{E_3}{2!}x^2 + \cdots \\
1 + y^2 &= 1 + E_0^2 + \left(\frac{E_0}{0!}\frac{E_1}{1!} + \frac{E_1}{1!}\frac{E_0}{0!}\right)x + \left(\frac{E_0}{0!}\frac{E_2}{2!} + \frac{E_1}{1!}\frac{E_1}{1!} + \frac{E_2}{2!}\frac{E_0}{0!}\right)x^2 + \cdots.
\end{aligned}$$

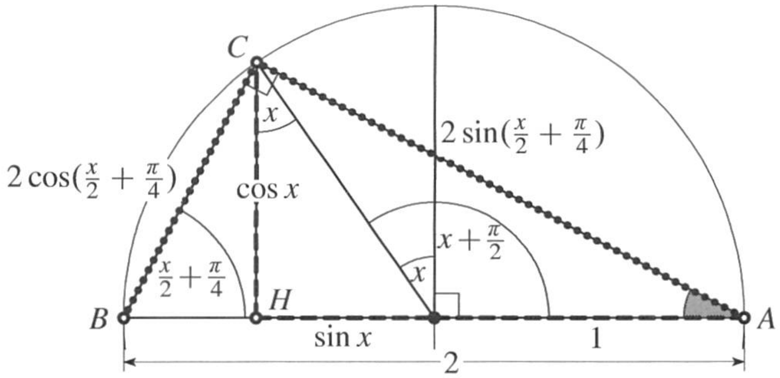



Figure 13 Proof of (36) from $AH/HC = AC/CB$ (Eucl. III.20 and Thales).

We first have from $y(0) = 1$ that $E_0 = 1$. Next we equalize the coefficients in the second and third row and obtain:

$$E_0 = 1\,, \quad E_1 = 1\,, \quad 2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k} \quad (n = 1, 2, 3, \ldots)\,. \tag{38}$$

This recursion for the Euler zigzag numbers was first discovered in 1879 by André by solving a question in combinatorics[17], as we shall see in the next section.

## 5 Connection with alternating permutations

> "Toutes les formules (...) étaient déjà connues; mais l'introduction du symbole $A_n$, moitié du nombre des permutations alternées de $n$ éléments distincts, leur donne, selon nous, un nouveau degré d'intérêt. [All formulas (...) were already known; but the introduction of the symbol $A_n$, half of the number of alternating permutations of $n$ distinct elements, gives them, in our opinion, a new degree of interest.]"
>
> (Désiré André, [2, p. 177])

**Alternating permutations.** On May 12, 1879 Désiré André [1] presented to the *Académie des Sciences* in Paris a "probablement toute nouvelle" notion in combinatorics, that of "permutation alternée" (today sometimes called *zigzag permutations*). A detailed paper was published two years later [2].

We call a permutation $\sigma : i \mapsto \sigma_i$ of $n$ objects $[n] = \{1, 2, \ldots, n\}$

with $\sigma_1 > \sigma_2 < \sigma_3 > < \cdots$ a *down-up (alternating) permutation,*
with $\sigma_1 < \sigma_2 > \sigma_3 <> \cdots$ an *up-down (alternating) permutation.*

It is easy to see that by the symmetry $\sigma_k \mapsto n+1-\sigma_k$ an down-up permutation becomes a up-down permutation and vice-versa. Therefore, for a given $n$, there are as many up-down as there are down-up permutations and André raised the question of finding this number $A_n$ (André's notation). It is convenient to set $A_0 = A_1 = 1$. For the next values, we see

[17] At that date, according to André, combinatorics was "une partie des Mathématiques où les méthodes étaient jusqu'à présent fort rares, pour ne pas dire inconnues [a part of mathematics where the methods were until now very rare, if not unknown]" (*Bull. Soc. Math. Fr.* 7, 1879, p. 63).

in Figure 14 that $A_2 = 1$, $A_3 = 2$, $A_4 = 5$. Comparing these numbers with the extended Bernoulli list (26), it is not difficult to guess a general theorem:

**André's Theorem** ([1, p. 965], [2, p. 170]). *The numbers $A_n$ counting the up-down or down-up alternating permutations of $n$ elements are, for all $n$, the Euler zigzag numbers* (26).

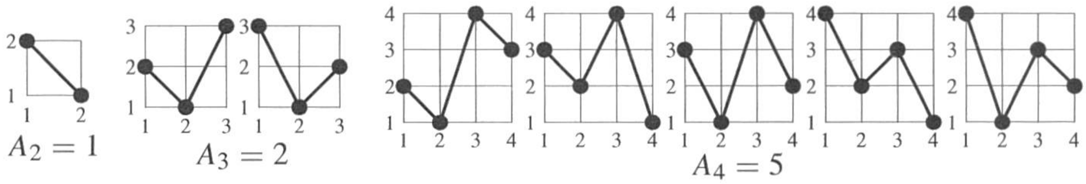



Figure 14 Down-up permutations for $n = 2, 3, 4$.

*Proof.* André supposes the $A_k$ to be known for $k \le n$ and searches all alternating permutations of $[n+1]$. Let $k+1$ be the position of the highest peak $\sigma_{k+1} = n+1$, the permutation going down on both sides. This peak will thus be followed, left and right from inside towards outside, by up-down permutations:

$$\underbrace{\square \ldots \square > \square}_{k} < \underset{k+1}{\overset{\sigma_{k+1}=n+1}{\square}} > \underbrace{\square < \square \ldots \square}_{n-k}.$$

There are $\binom{n}{k}$ possibilities for partitioning the remaining $n$ points $[n]$ in two sets with $k$ and $n-k$ elements and $A_k$, respectively $A_{n-k}$, possibilities to arrange these elements as up-down permutations. Adding up $\binom{n}{k} A_k A_{n-k}$ for all $k$, we so obtain *all* alternating permutations of $[n+1]$, hence the *double* of $A_{n+1}$. This way of counting is precisely the recursion formula (38) for the Euler zigzag numbers. $\square$

From here, going through the above proofs backwards, André arrived[18] at formulas (37) and then to (36), the e. g. f. $\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)$ of the $A_k$ ("notre formule fondamentale"). Finally, in the search for the simplest recursions, he developed a list of formulas, among which are (29) and (32). Some years later, he discovered formula (33) by determining the asymptotic value of the probability for a permutation to be an alternating one[19].

**Entringer's Lemma.** Unaware of André's work, Aubrey J. Kempner (1880–1973) [19] and Roger C. Entringer [8] rediscovered the alternating permutations in 1933 and 1966 respectively. While André focussed on the position of the highest peak, Entringer focussed *on the value of* $\sigma_1$. So we define the *Entringer numbers*:

$$\begin{aligned} E_{nk} &= \text{\# of down-up alt. perm. of } \{1, 2, \ldots, n+1\} \text{ with } \sigma_1 = k+1 \\ &= \text{\# of up-down alt. perm. of } \{1, 2, \ldots, n+1\} \text{ with } \sigma_1 = n+1-k\,. \end{aligned}$$

[18] "It's a very nice exercise" according to Arnol'd [5, p. 64].
[19] *Comptes rendus* 97, 1883, pp. 983–984.

We see in Figure 14 that the first of these numbers are $E_{11} = 1$, $E_{21} = 1$, $E_{22} = 1$, $E_{31} = 1$, $E_{32} = 2$, $E_{33} = 2$. In general, they satisfy *Entringer's Lemma* [8, p. 242]:

$$E_{nk} = E_{n,k-1} + E_{n-1,n-k} \,. \tag{39}$$

*Proof of Entringer's Lemma.* We simplify Entringer's proof by observing that there are two types of reductions for alternating permutations. We are dealing with the case of down-up permutations, the other case being similar.

1. The down-up permutations in $E_{nk}$ such that $\sigma_2 = \sigma_1 - 1$ are, by removing $\sigma_1$, one-to-one with the up-down permutations in $E_{n-1,n-k}$ (gray arrows; Fig. 15, left).

2. The remaining down-up permutations in $E_{nk}$ are, after exchanging the rows $k+1 \leftrightarrow k$, one-to-one with the down-up permutations in $E_{n,k-1}$ (black arrows; Fig. 15, right). □

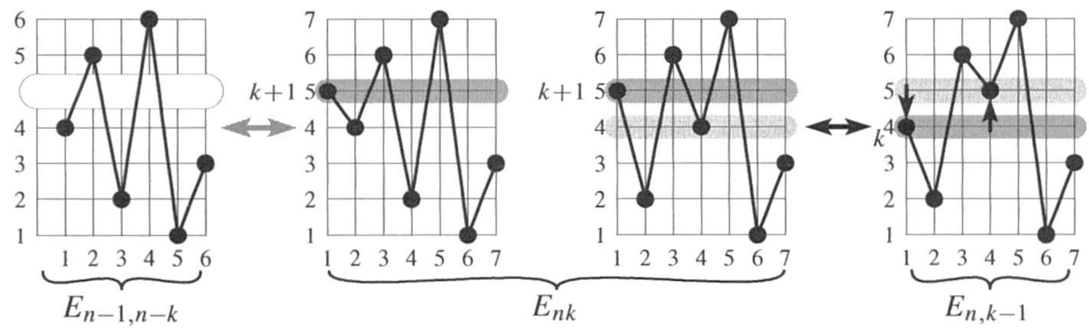



Figure 15 Proof of Entringer's Lemma ($n = 6$, $k = 4$).

**The Seidel–Entringer–Arnol'd (SEA-) triangle.** Starting from $E_{00} = 1$ and $E_{n0} = 0$ ($n = 1, 2, 3, \ldots$), formula (39) represents an elegant algorithm for computing all these numbers (see Fig. 16). Since in this formula the indices of the second term (for $n-1$) run in the opposite direction, it is nicer to arrange them in a triangle as upwards-downwards alternating columns: each $E_{nk}$ is then the sum of its neighbor above (or below) and the one to the left, exactly as in the Bürgi–Bernoulli algorithm. This triangle was discovered by Philipp Ludwig von Seidel[20] (1821–1896) and by Vladimir Arnol'd[21] (1937–2010).

Applying the two bijections of the previous proof repeatedly, we fill the nodes of the entire SEA-triangle with the corresponding alternating permutations and we obtain Figure 17. In this picture the permutations of the first type are, along the gray arrows, one-to-one with all permutations in $E_{n-1,n-k}$ of "opposite downing mode". They are placed inside a dotted region.

[20]For Seidel (1821–1896) it was an excellent means for calculating the Bernoulli numbers ("(...) in welchem sich wahrscheinlich die einfachste Genesis der B e r n o u l l i ' s c h e n Z a h l e n ausspricht" [29, p. 158]). Seidel's work remained unnoticed until Dominique Dumont (1947–2007) studied anew the "Euler–Seidel matrices" (*Séminaire Lotharingien de Combinatoire* B05c, 1981).

[21]"To calculate the Euler and Bernoulli numbers quickly, it is convenient to use the classical Euler–Bernoulli triangle, similar to the Pascal triangle" [4, p. 3]. "J'appelle [ce] triangle de Euler–Bernoulli parce que Pascal ne l'a pas considéré, et parce que Euler et Bernoulli ne l'ont pas considéré non plus [I call this triangle Euler–Bernoulli triangle because Pascal did not consider it, and because Euler and Bernoulli did not either consider it]" [5, p. 63].

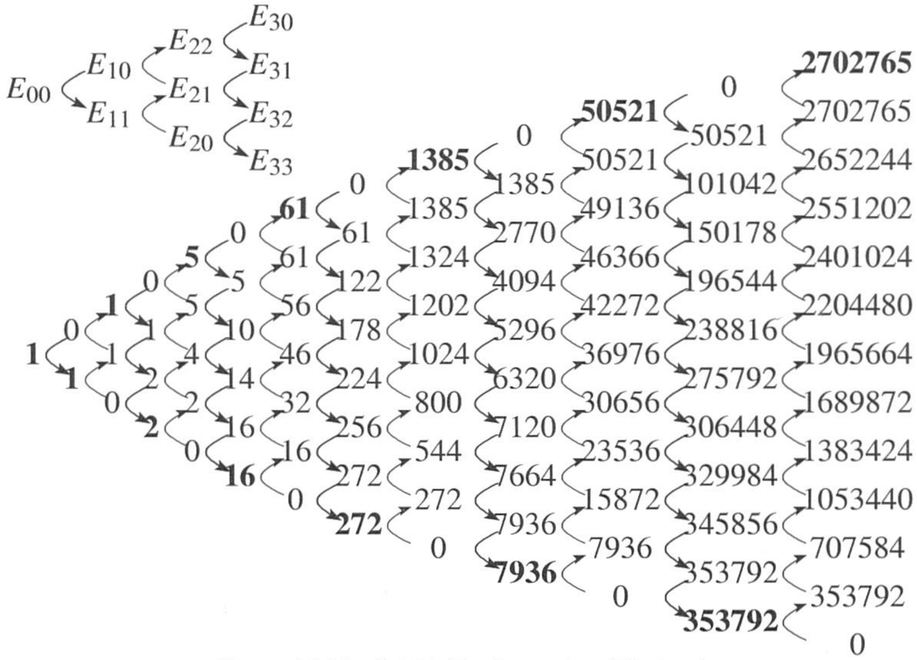


Figure 16 The Seidel–Entringer–Arnol'd triangle.

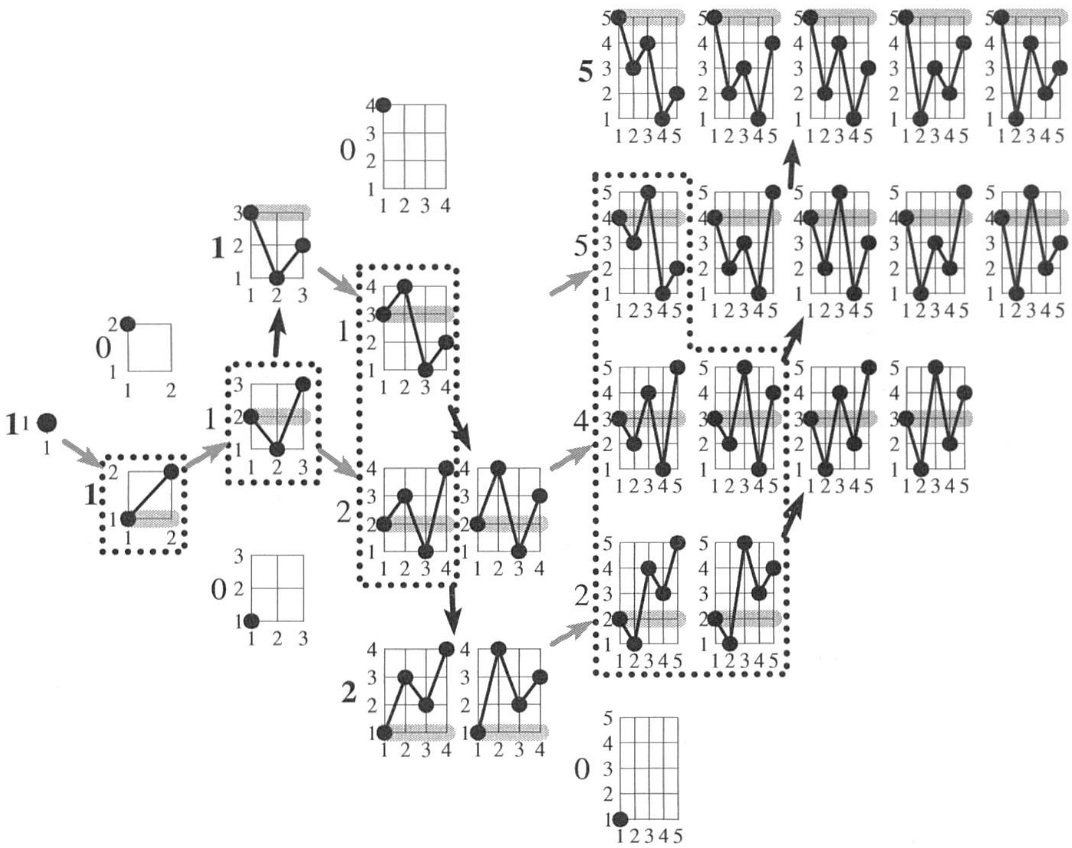


Figure 17 Filling the nodes of the SEA-triangle for $n = 0, 1, 2, 3, 4$ with its corresponding alternating permutations.

**Euler and Bernoulli numbers.** By iterating (39) we have

$$E_{nn} = \sum_{k=0}^{n-1} E_{n-1,k} = \text{(André's theorem)} = E_n \,. \tag{40}$$

This means that *at the end of each column in the SEA-triangle appear the Euler zigzag numbers, the Euler numbers above, the Bernoulli numbers below* (*see* (31))[22]. This can also be seen in Figure 17 where the number of dotted permutations in column $n$, which is equal to $E_{nn}$, is equal to the number of *all* permutations in the previous column.

**The Boustrophedon Transform.** If the simple algorithm (39) already produces such interesting sequences of numbers as Euler's or Bernoulli's, one might ask which other interesting sequences of numbers $(b_n)$ appear at the end of each column in the SEA-triangle if, instead of $e_0 = (1, 0, \ldots)$, we feed this triangle with an arbitrary sequence $(a_n)$? Figure 18 illustrates this question originally formulated by Richard Guy about this "boustrophedon" (or "ox-plowing") transform $\mathcal{B} : (a_n) \mapsto (b_n)$. It was solved in 1996:

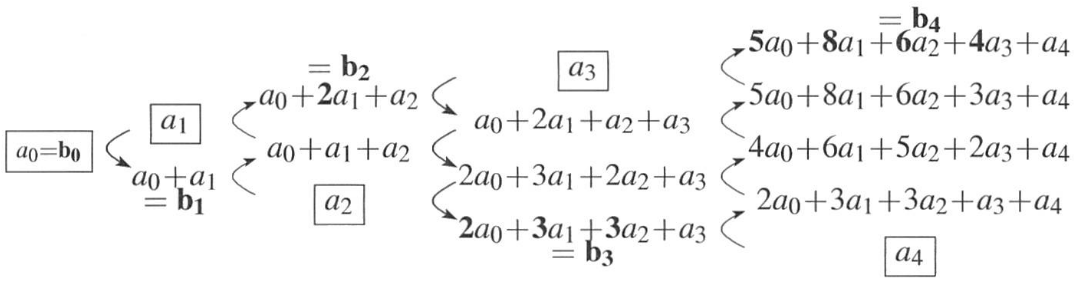



Figure 18 The Boustrophedon transform $(b_n) = \mathcal{B}(a_n)$.

**The Boustrophedon Theorem** (Millar, Sloane and Young, [26, Thm. 1]).

$$b_n = \sum_{k=0}^{n} \binom{n}{k} E_{n-k}\, a_k \,. \tag{41}$$

A "sketch of proof" based on paths in a directed graph and so-called "box diagrams" is given in [26]. We understand it here more easily as follows:

*Proof.* Using linear superposition, we separate the coefficients of $a_0$ in Figure 18, which are the entries of the SEA-triangle of Figure 16, and those of $a_1$, $a_2$, $a_3$, etc. which are the entries obtained by calculating the values of the image sequences of $e_1 = (0, 1, 0 \ldots)$, $e_2 = (0, 0, 1, 0, \ldots)$, $e_3 = (0, 0, 0, 1, 0, \ldots)$, etc. which we denote by $(c_j^1)$, $(c_j^2)$, $(c_j^3)$, etc. We so obtain modified SEA-triangles, constructed by the same rules, but with different initial values (see Fig. 19).

[22] Émile Picard (1856–1941) writes that "les nombres $A_n$ de Désiré André remplaceront sans doute quelque jour en analyse les nombres de Bernoulli et les nombres d'Euler [the numbers $A_n$ of Désiré André will without doubt some days replace in analysis the Bernoulli and Euler numbers]" (Rapport sur les travaux de M. Désiré André, 1910, Archives de l'Académie des sciences). Entringer, unaware of André's work, proved the appearence of these numbers with a lot of analytic calculations [8].

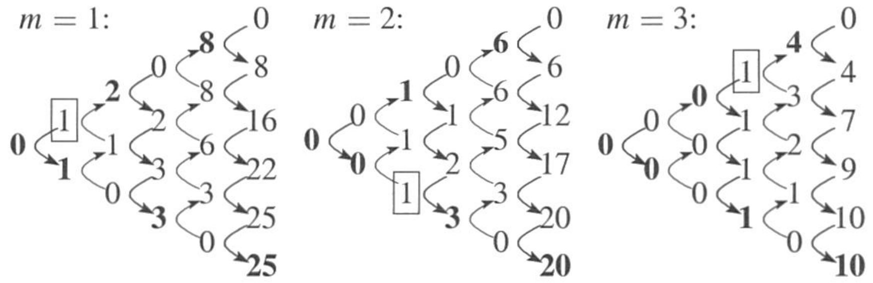



Figure 19 The images $\mathcal{B}(e_m)$ for $m = 1, 2, 3$.

We have $\mathcal{B}(e_0) = (c_j^0)$ with $c_j^0 = E_j$ and (41) means that at the end of each column appear Euler zigzag numbers, shifted and multiplied by a binomial coefficient:

| $m\|j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | $E_0{=}1$ | $E_1{=}1$ | $E_2{=}1$ | $E_3{=}2$ | $E_4{=}5$ | $E_5{=}16$ | $E_6{=}61$ | $E_7{=}272$ |
| | | $\searrow\cdot 1$ | $\searrow\cdot 2$ | $\searrow\cdot 3$ | $\searrow\cdot 4$ | $\searrow\cdot 5$ | $\searrow\cdot 6$ | $\searrow\cdot 7$ |
| 1 | 0 | 1 | 2 | 3 | 8 | 25 | 96 | 427 |
| | | | $\searrow\cdot \frac{2}{2}$ | $\searrow\cdot \frac{3}{2}$ | $\searrow\cdot \frac{4}{2}$ | $\searrow\cdot \frac{5}{2}$ | $\searrow\cdot \frac{6}{2}$ | $\searrow\cdot \frac{7}{2}$ |
| 2 | 0 | 0 | 1 | 3 | 6 | 20 | 75 | 336 |
| | | | | $\searrow\cdot \frac{3}{3}$ | $\searrow\cdot \frac{4}{3}$ | $\searrow\cdot \frac{5}{3}$ | $\searrow\cdot \frac{6}{3}$ | $\searrow\cdot \frac{7}{3}$ |
| 3 | 0 | 0 | 0 | 1 | 4 | 10 | 40 | 175 |

Therefore it is enough to prove that for $(c_j^m) = \mathcal{B}(e_m)$ $(m = 1, 2, 3\ldots)$ we have

$$c_j^m = \binom{j}{m} E_{j-m} \quad \text{for} \quad j \geq m. \tag{42}$$

The key to proving equality (42) is to replace alternating permutations of $[n+1]$ by *alternating injective maps into a larger image set*:

$$\begin{aligned} E_{n+m,k}^{(m)} &= \text{\# of down-up alternating injections } \{1,\ldots,n+1\} \to \{1,\ldots,n+1+m\} \\ &\qquad\qquad \text{with } \sigma_1 = k+1 \\ &= \text{\# of up-down alternating injections } \{1,\ldots,n+1\} \to \{1,\ldots,n+1+m\} \\ &\qquad\qquad \text{with } \sigma_1 = n+1+m-k\,. \end{aligned}$$

*These numbers, for $m = 1, 2, 3$ respectively, are precisely the entries of the triangles in Figure* 19. Indeed, for example with $m = 1$, we construct Figure 20 analogously to Figure 17. We observe that, in each column, all injections with the same initial value, after removing this value, are built from the previous ones (according to the direction of the arrows) in the previous column. Therefore, by construction, we again have the recursion formula (39).

Now fix $m$ and consider all the alternating injective maps at the top or at the bottom (according to the downing mode) of a column. This number can be written $E_{n+m,n+m}^{(m)}$ and represent maps from $[n+1]$ to $[n+1+m]$. Since for all such maps $\sigma_1$ is fixed at the top

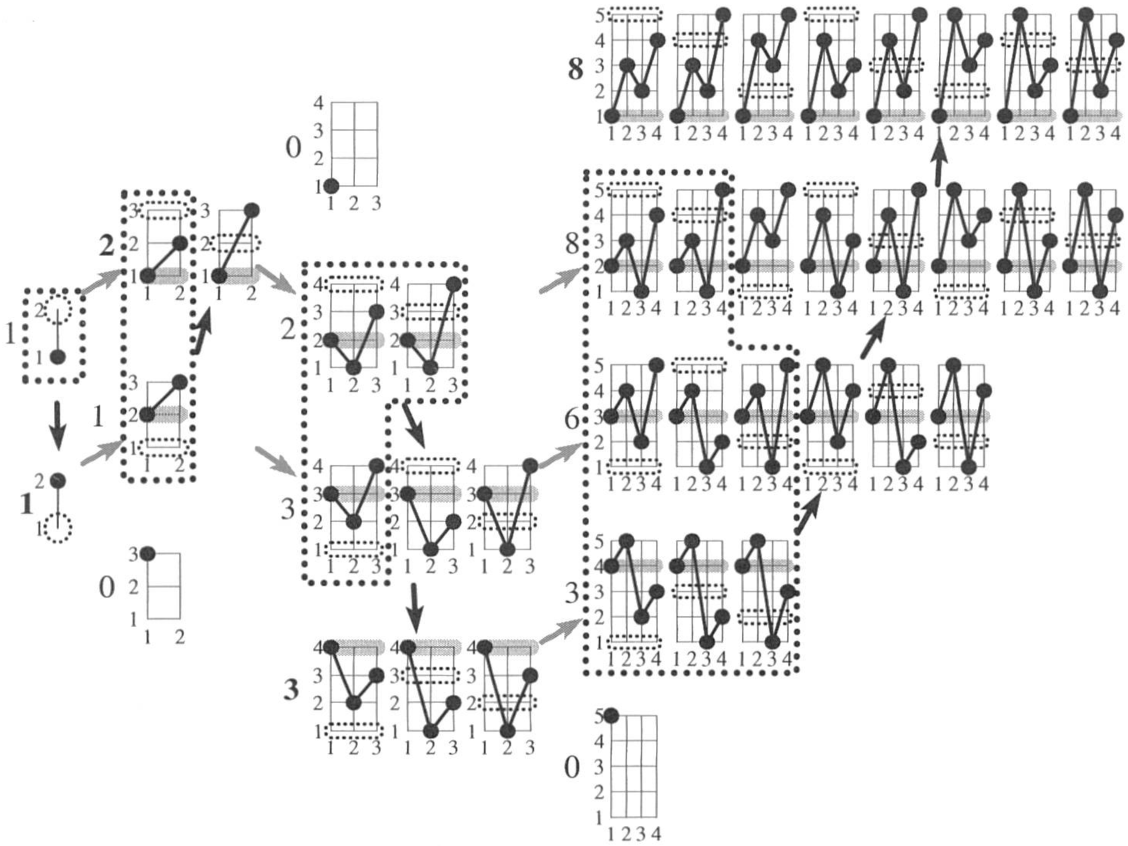


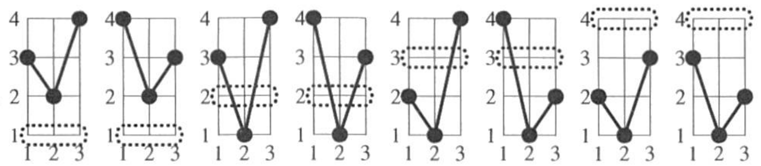
Figure 20 Modified Entringer numbers $E^{(1)}_{n+1,k}$ and maps for $n+1 = 1, 2, 3, 4$ arranged as SEA-triangle (missing image values are surrounded by dotted ovals).

or at the bottom line, we are free to move: we thus have $\binom{n+m}{m}$ possibilities to choose $m$ values to avoid among the $n+m$ possibilities in the image set and we have $E_n$ possibilities to choose an alternating permutation of the $n$ remaining ones. So we have

$$E^{(m)}_{n+m,n+m} = \binom{n+m}{m} \cdot E_n \quad \text{and thus} \quad c^m_j = \binom{j}{m} \cdot E_{j-m} \quad (j \geq m)$$

because $j - m = n$.

For example, if $n = 3$ and $m = 1$, after removing $\sigma_1$, the **8** alternating injections on the top of the fourth column in Figure 20 can be arranged according to the $\binom{4}{1} = 4$ values not in the image set and the $E_3 = 2$ down-up permutations of the remaining values:



□

*Remark.* The Boustrophedon Theorem easily gives the link between the corresponding e. g. f. $A$, $B$ of the sequences $(a_n)$, $(b_n)$. Dividing both sides of (41) by $n!$, the expression obtained for $b_n/n!$ is the same as that in the product

$$\underbrace{\left(b_0 + b_1 x + \tfrac{b_2}{2!}x^2 + \cdots\right)}_{=B(x)} = \underbrace{\left(E_0 + E_1 x + \tfrac{E_2}{2!}x^2 + \cdots\right)}_{=E(x)=\sec x + \tan x}\underbrace{\left(a_0 + a_1 x + \tfrac{a_2}{2!}x^2 + \cdots\right)}_{=A(x)}. \tag{43}$$

Therefore, *the Boustrophedon transform multiplies the corresponding e. g. f. by $E(x)$*. Otherwise, (42) shows that for $j \geq m$ the $j$th term of the e. g. f. of the sequence $\mathcal{B}(e_m)$ is

$$\binom{j}{m} E_{j-m} \cdot \frac{x^j}{j!} = \frac{x^m}{m!} \cdot E_{j-m} \frac{x^{j-m}}{(j-m)!}.$$

Therefore, by adding up, we see that the e. g. f. of $\mathcal{B}(e_m)$ is $E(x)\frac{x^m}{m!}$ and we obtain (43) by linear superposition.

Differentiating (37) we get $2y'' = 2y' \cdot y$, i.e., $E'' = E \cdot E'$, hence (43) shows that $\mathcal{B}(E_1, E_2, E_3, \ldots) = (E_2, E_3, E_4, \ldots)$. This can also be seen in Figure 16 by deleting the top and the bottom side of the triangle. Furthermore, by deleting only the top (respectively the bottom) side of the triangle, we see that

$$\mathcal{B}(E_0, 0, E_2, 0, E_4, \ldots) = (E_1, E_2, E_3, \ldots)$$

and

$$\mathcal{B}(0, E_1, 0, E_3, \ldots) = (0, E_2, E_3, \ldots).$$

**Paths in a directed graph.** In 1991, Arnol'd [3, p. 542] proposed another interpretation of the Entringer numbers. He observed that (39) associates a directed graph $\Gamma$ with the SEA-triangle (see Fig. 21, left): if we concentrate all maps representing $E_{nk}$ in Figure 17 or $E^{(1)}_{n+1,k}$ in Figure 20 to a single node $\bullet_{nk}$ by keeping the arrows, we always obtain a part of this same graph $\Gamma$.

If now we choose the starting node $\bullet_{00}$ and ask for the number of different paths starting in $\bullet_{00}$ and ending in one of the nodes $\bullet_{nk}$ we have

$$E_{nk} = \#\,\textit{paths from } \bullet_{00} \textit{ to } \bullet_{nk}$$

$$\text{and, in particular,} \quad E_{nn} = E_n = \#\,\textit{paths from } \bullet_{00} \textit{ to } \bullet_{nn}.$$

For example, we see in Figure 21 (right) that there are $E_{42} = 4$ paths starting in $\bullet_{00}$ and ending in $\bullet_{42}$: these $4 = 2 + 2$ paths are built by the $2 = E_{32} = E_{41}$ paths ending in the incoming nodes $\bullet_{32}$ and $\bullet_{41}$.

Moreover, we observe that by remembering the creation of the paths, *we have a natural bijection between the paths in $\Gamma$ from $\bullet_{00}$ to $\bullet_{nk}$ and the alternating permutations in $E_{nk}$*. For example, the four paths above correspond, from top to bottom and left to right, to the four permutations in $E_{42}$ from left to right in Figure 17.

In the same way, for $m \geq 1$, we have

$$E^{(m)}_{n+m,k} = \#\,\textit{paths from } \bullet_{m0} \textit{ to } \bullet_{n+m,k}$$

and, in particular,

$$E^{(m)}_{n+m,n+m} = \binom{n+m}{m} E_n = \binom{n+m}{n} E_n = \#\,\textit{paths from } \bullet_{m0} \textit{ to } \bullet_{n+m,n+m}. \tag{44}$$

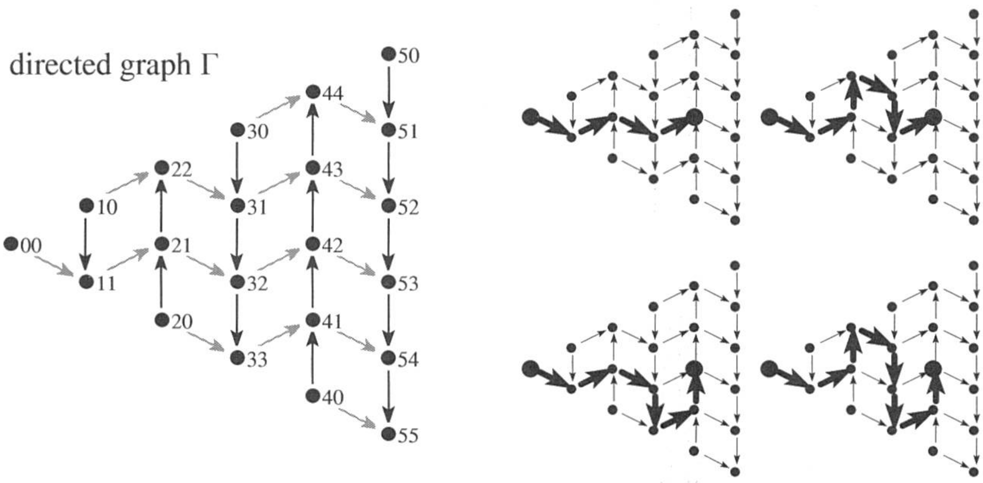



Figure 21 The directed graph Γ (left); paths from $\bullet_{00}$ to $\bullet_{42}$ (right).

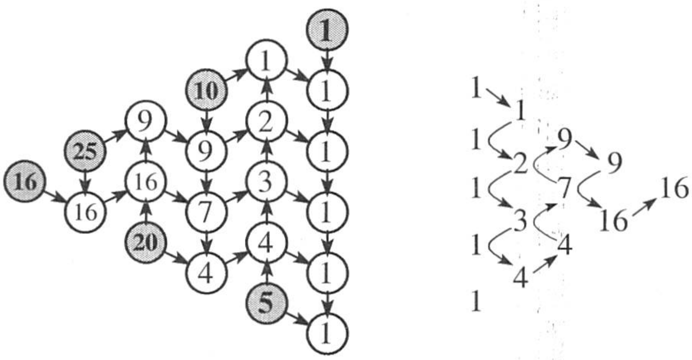


Figure 22 # of paths ending in $\bullet_{55}$ (left).

**Paths ending in a given node.** We now choose a fixed node, for example $\bullet_{55}$, and ask for the number of paths starting in $\bullet_{nk}$ and ending in $\bullet_{55}$. The result is displayed in Figure 22 (left). This time, we compute these values from right to left. The first column is filled by 1's. Then the value of any further node, for example $\bullet_{32}$, is the sum of possible paths from the outgoing nodes $\bullet_{33}$ and $\bullet_{42}$, hence $4+3=7$. Using (44), we understand why the dark nodes in Figure 22 bear the values

$$5 = E_{55}^{(4)} = \binom{5}{1} E_1, \quad 10 = E_{55}^{(3)} = \binom{5}{2} E_2, \quad \ldots, \quad 16 = E_{55}^{(0)} = \binom{5}{5} E_5 . \qquad (45)$$

**"Discretizing" Johann's iterated involutes.** Suppose to have a fixed angle $b$ and a fixed integer $n$. We explain the idea of the construction[23] in Figure 23 for the case $n = 5$. We fill the circular sector of radius 1 and angle $b$ by 5 isosceles triangles of side lengths 1, 1 and $s = 2\sin\frac{b}{10}$. We then construct a discrete involute to this discrete arc by attaching

[23] This construction was motivated by an idea of Y.S. Chaikovsky [15] who obtained in a similar manner the Taylor series for sinus and cosinus. In his paper, the anologue of the SEA-triangle is the Pascal's triangle.

4 triangles, all similar to the original triangles, of base lengths $s^2$, $2s^2$, $3s^2$, $4s^2$ (see the triangle to the left of Fig. 23). For the next involute, we have, for example,

$$ED = s \cdot DA = s \cdot (CB + BA) = s \cdot (4s^2 + 3s^2) = 7s^3 .$$

The integer coefficients of all these arc lengths are computed by the triangle to the right of Figure 22. These numbers are the same as those in the white circles to the left of the same figure, because they are formed by the same alternating addition process. The coefficients of the total arc lengths for these involutes are the sums of the columns, i.e., the values in the dark circles given by (45).

The same is true for any $n$ (see Fig. 24) and, if $n \to \infty$, *the arc length of the $i$th involute tends, for each $i$, to*

$$E_i \cdot \binom{n}{i} s^i \overset{s \to \frac{b}{n}}{\longrightarrow} E_i \cdot 1 \cdot \big(1 - \tfrac{1}{n}\big)\big(1 - \tfrac{2}{n}\big) \cdots \big(1 - \tfrac{i-1}{n}\big) \cdot \frac{b^i}{i!} \overset{n \to \infty}{\longrightarrow} E_i \cdot \frac{b^i}{i!} .$$

We have thus obtained a combinatoric proof of Johann's arc lengths in Figure 6. Figure 24 becomes, for $n \to \infty$, the picture of Figure 11, which was the origin of the beautiful access to the series for sec $b$ and tan $b$.

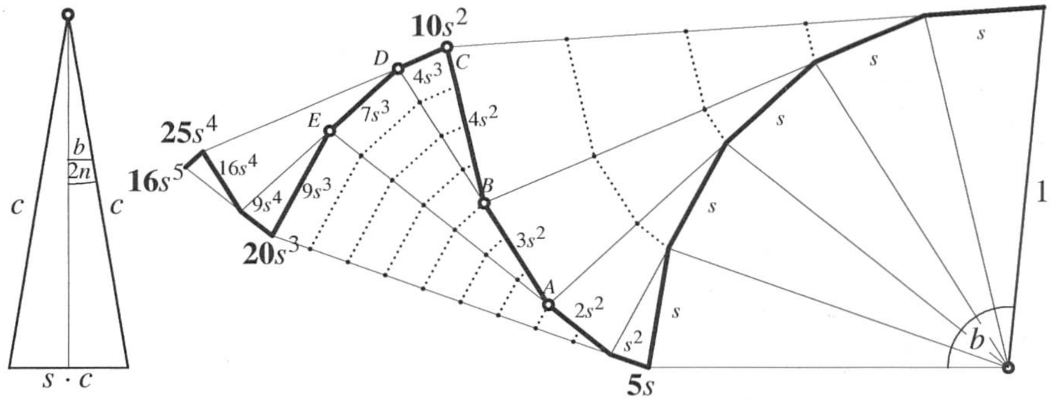



Figure 23 Discrete Bernoulli's algorithm for $n = 5$, $b = \frac{\pi}{2} + 0.1$.

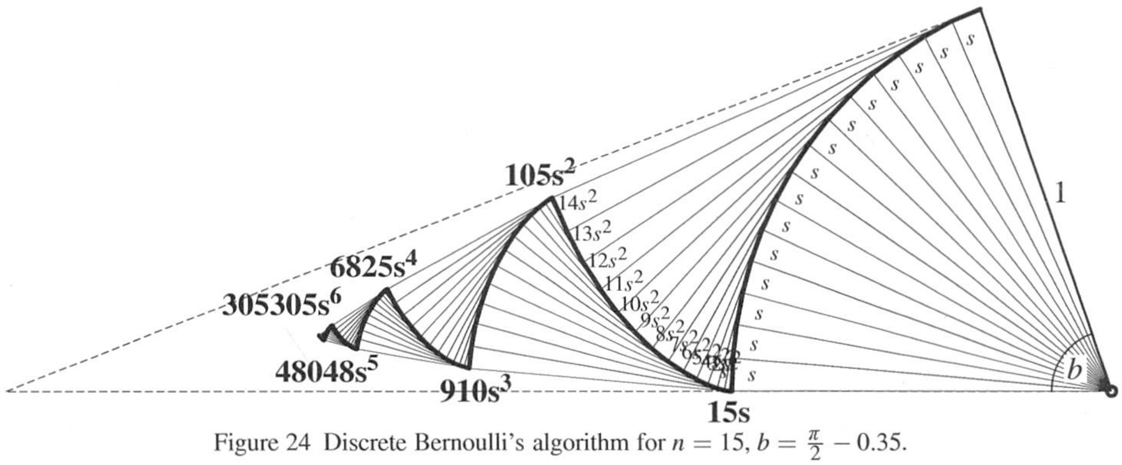



Figure 24 Discrete Bernoulli's algorithm for $n = 15$, $b = \frac{\pi}{2} - 0.35$.

**Acknowledgment.** We have the duty to thank E. Delucchi (Fribourg) for his talk *Aspetti combinatorici dell'opera di Jakob Bernoulli* (Ascona, Sept. 2016). During his presentation we noticed the connection between the SEA-triangle and Johann's work, which motivated the last part of this article. It is also a pleasure to thank R. Chan (Auckland), M. Anderegg (Geneva) for many helpful remarks and our respective wives for their patience during the redaction of this paper.

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Philippe Henry
33 ch. de La-Californie
CH-1222 Vésenaz
philippe_henry@bluewin.ch
philippe.henry@a3.epfl.ch

Gerhard Wanner
37 ch. des Pinsons
CH-1226 Thonex
gerhard.wanner@unige.ch