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Another Wolstenholme-type congruence

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1 Introduction

If in 1862 J. Wolstenholme [13] proved that *the numerator of the fraction* $1+\frac{1}{2}+\cdots+\frac{1}{n-1}$ *when reduced to its lowest terms* [*for a prime* $n>3$] *is divisible by* n^2, today one simply considers each term of the sum as the inverse of an element of $\mathbb{Z}_{n^2}$ and sets $\sum_{i=1}^{n-1}\frac{1}{i}\equiv 0 \pmod{n^2}$. The viewpoint has changed but the fascination remains intact. Combining the previous result with the fact that $\sum_{i=1}^{n-1}\frac{1}{i^2}\equiv 0 \pmod{n}$ allowed him to prove $\binom{2n-1}{n-1}\equiv 1 \pmod{n^3}$. At the dawn of the XX$^{\text{th}}$ century, J.W.L. Glaisher [3, 4] extended Wolstenholme's theorem, $\binom{2p-1}{p-1}\equiv 1 \pmod{p^3}$ to $\mathbb{Z}_{p^4}$ for primes $p>3$ and connected it to

Der sogenannte *erste Fall des Satzes von Fermat* besagt, dass $x^p+y^p\neq z^p$ für Primzahlen p mit $p\nmid xyz$ gilt. Bereits im Jahre 1847 realisierte Cauchy (siehe [12, p. 155]), dass $\sum_{i=1}^{(p-1)/2} i^{p-4}\equiv 0 \pmod{p}$ folgt, wenn dieser *erste Fall* für eine Primzahl p nicht gilt. Diese Summe wiederum ist über den kleinen Satz von Fermat verknüpft mit den Bernoulli-Zahlen und einem Resultat von Genocchi: $\sum_{i=1}^{(p-1)/2}\frac{1}{i^3}\equiv -2B_{p-3} \pmod{p}$. Derartige Summen wurden zu einer Quelle der Inspiration für Sylvester, Wolstenholme, Morley, Glaisher, Mirimanoff, Vandiver und Lehmer. In der vorliegenden Arbeit zeigt der Autor

$$\sum_{\substack{0<i<j<p\\ i \text{ odd},\, j \text{ even}}}\frac{1}{ij}\equiv -\frac{7}{48}pB_{p-3} \pmod{p^2}.$$

Eine Version dieser Summe modulo p war vor Kurzem die Grundlage für einen elementaren Beweis der Kongruenz von Morley.

Bernoulli numbers by obtaining $\binom{2p-1}{p-1} \equiv 1 - \frac{2}{3}p^3 B_{p-3} \pmod{p^4}$. More than half a century later, Selfridge and Pollack [10] identified the first irregular prime p dividing B_{p-3}. Another thirty years after, J. McIntosh [10] defined *Wolstenholme primes* as verifying one of the three equivalent conditions:

i) $\displaystyle\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}$

ii) $p | B_{p-3}$

iii) $\displaystyle\sum_{1 \le k \le \lfloor \frac{p}{6} \rfloor + 1}^{\lfloor \frac{p}{4} \rfloor} \frac{1}{k^3} \equiv 0 \pmod{p}$

and thanks to the third condition confirmed the sole existence of two Wolstenholme primes smaller than $2 \cdot 10^8$. He also conjectured an infinite number of such primes for probabilistic reasons. Since then, no other Wolstenholme prime has been identified. A detailed account of the history of Wolstenholme-type congruences from the XIX$^{\text{th}}$ to the XXI$^{\text{th}}$ century is contained in [11].

Recently, an elementary proof of Morley's congruence theorem [1], $4^{p-1} \equiv \pm\binom{p-1}{\frac{p-1}{2}} \pmod{p^3}$, was found depending on the fact that

$$\sum_{\substack{0<i<j<p \\ i \text{ odd}, j \text{ even}}} \frac{1}{ij} \equiv 0 \pmod{p} \quad \text{for primes } p > 3.$$

Our aim is to prove that

$$\sum_{\substack{0<i<j<p \\ i \text{ odd}, j \text{ even}}} \frac{1}{ij} \equiv -\frac{7}{48} p B_{p-3} \pmod{p^2}.$$

For example, in the case $p = 7$, one has

$$\begin{aligned} \frac{1}{1\cdot 2} + \frac{1}{1\cdot 4} + \frac{1}{1\cdot 6} + \frac{1}{3\cdot 4} + \frac{1}{3\cdot 6} + \frac{1}{5\cdot 6} &\equiv 25 + 37 + 41 + 45 + 30 + 18 \\ &= 196 \equiv 0 \pmod{7^2}. \end{aligned}$$

Our presentation relies almost exclusively on classical properties of Bernoulli numbers and polynomials from [8, Ch. 15] and [5] that we recall below.

2 Prerequisites

If we define in the standard way

$$S_m(p) := \sum_{i=1}^{p-1} i^m \tag{1}$$

then we have

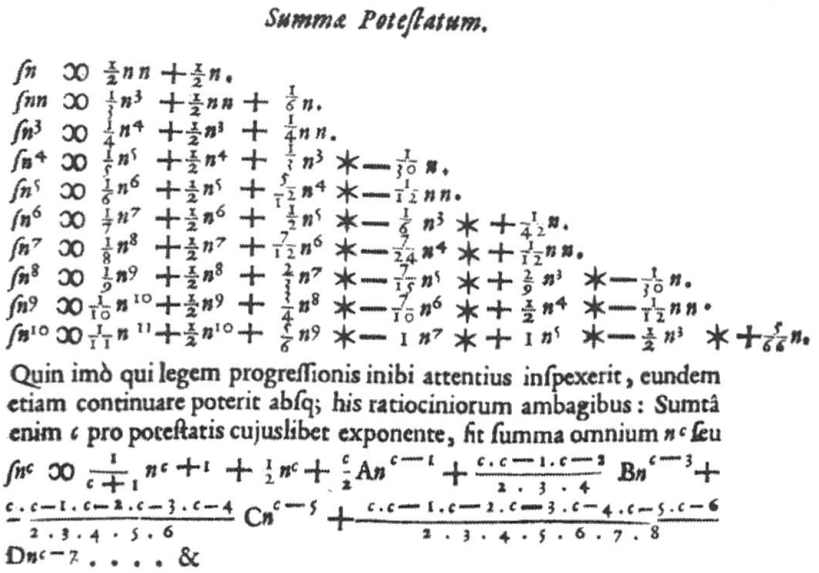
$$(m+1)S_m(p) = B_{m+1}(p) - B_{m+1} = \sum_{i=0}^{m+1} \binom{m+1}{i} p^{m+1-i} B_i, \tag{2}$$

where the B_m are the mth Bernoulli numbers given by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k,$$

and $B_m(x) := \sum_{k=0}^{m} \binom{m}{k} B_k x^{m-k}$ is the mth Bernoulli polynomial, which verifies the particular property

$$B_n\left(\frac{1}{2}\right) = (2^{1-n} - 1)B_n. \tag{3}$$


Summa Poteſtatum.

Quin imò qui legem progreſſionis inibi attentius inſpexerit, eundem etiam continuare poterit abſq; his ratiociniorum ambagibus: Sumtâ enim c pro poteſtatis cujuslibet exponente, fit ſumma omnium n^c ſeu

Figure 1 Jacob Bernoulli, *Ars Conjectandi*, 1713
With permission of Bibliotheque de Genève, Kc152, p. 97

3 Four progressive lemmas

We will start by recalling a very particular case of Leudesdorf's theorem [2, 5], that we prove for completeness.

Lemma 1. *If p is prime greater than 3, n is even and $p - 1 \nmid n$ then*

$$\sum_{1 \le i \le p-1} \frac{1}{i^n} \equiv \sum_{1 \le i \le \frac{p-1}{2}} \frac{1}{i^n} \equiv 0 \pmod{p}. \tag{4}$$

Proof. Let g denote a generator of $\mathbb{Z}_p^*$. Then

$$\sum_{1\leq i\leq p-1}\frac{1}{i^n}\equiv\sum_{1\leq i\leq p-1}\frac{1}{(gi)^n}\equiv\frac{1}{g^n}\sum_{1\leq i\leq p-1}\frac{1}{i^n}\equiv 0 \pmod p,$$

since $1/g^n \not\equiv 1 \pmod p$. If n is even then $i^n \equiv (p-i)^n \pmod p$ and so the second equivalence of (4) follows. □

Our second equivalence was already known to Genocchi in 1852 [12, p. 121] and generalized extensively in 2000 by Zhi-Hong Sun [14, Theorem 5.2.].

Lemma 2. *If p is prime and $p > 3$ then*

$$\sum_{0<i\leq\frac{p-1}{2}}\frac{1}{i^3}\equiv -2B_{p-3} \pmod p. \tag{5}$$

Proof. Working in $\mathbb{Z}_p$ we transform the summand in (5) into an expression of the form (1) by applying Fermat's little theorem, and follow up by the identities (2), (3) and (4):

$$\begin{aligned}\sum_{0<i\leq\frac{p-1}{2}}\frac{1}{i^3}&\equiv\sum_{0<i\leq\frac{p-1}{2}}i^{p-4}=S_{p-4}(\frac{p+1}{2})=\frac{1}{p-3}\left(B_{p-3}(\frac{p+1}{2})-B_{p-3}\right)\\&\equiv\frac{-1}{3}\left(B_{p-3}(\frac{1}{2})-B_{p-3}\right)\equiv\frac{-1}{3}\left((2^{4-p}-1)B_{p-3}-B_{p-3}\right)\equiv -2B_{p-3}.\end{aligned}$$ □

The next congruence is proposed as an exercise by Hao Pan in [7].

Lemma 3. *If p is prime and $p > 3$ then*

$$\sum_{\substack{0<i<j<p\\ j\text{ even}}}\frac{1}{i^2j}\equiv -\frac{3}{8}B_{p-3} \pmod p. \tag{6}$$

Proof. We proceed as above

$$\begin{aligned}\sum_{\substack{0<i<j<p\\ j\text{ even}}}\frac{1}{i^2j}&\equiv\sum_{\substack{0<i<j<p\\ j\text{ even}}}\frac{i^{p-3}}{j}=\sum_{\substack{0<j<p\\ j\text{ even}}}\frac{S_{p-3}(j)}{j}=\sum_{\substack{0<j<p\\ j\text{ even}}}\frac{B_{p-2}(j)-B_{p-2}}{j(p-2)}\\&\equiv-\sum_{\substack{0<j<p\\ j\text{ even}}}\frac{1}{2j}\sum_{k=0}^{p-2}\binom{p-2}{k}B_k j^{p-2-k}-B_{p-2}\\&\equiv-\frac{1}{2}\sum_{k=0}^{p-3}\sum_{\substack{0<j<p\\ j\text{ even}}}\binom{p-2}{k}B_k j^{p-3-k}\qquad\text{let } j=2m\end{aligned}$$

$$\equiv -\frac{1}{2}\sum_{k=0}^{p-3}\binom{p-2}{k}B_k\sum_{m=1}^{\frac{p-1}{2}}\frac{1}{(2m)^{k+2}}.$$

If $1 < k < p-3$ and k is odd then $B_k = 0$. Moreover if k is even, Leudesdorf's congruence (4) implies the last summand is 0. We are therefore left with the two terms $k = 1$ and $k = p-3$ to examine.

For $k = 1$ we get $-\frac{1}{2}(-2)\frac{-1}{2}\frac{1}{8}(-2B_{p-3}) = \frac{1}{8}B_{p-3}$ by using (5).

For $k = p-3$ we get $-\frac{1}{2}(-2)B_{p-3}\frac{-1}{2} = -\frac{1}{2}B_{p-3}$ by using Fermat.

Hence $\frac{1}{8}B_{p-3} + (-\frac{1}{2}B_{p-3}) = -\frac{3}{8}B_{p-3}$ gives the desired result. □

Our final lemma follows the same approach as the two preceding ones, but its proof requires all three conditions (3), (4) and (5).

Lemma 4. *If p is prime and $p > 3$ then*

$$\sum_{\substack{0<i<j<p\\ i \text{ and } j \text{ even}}}\frac{1}{i^2j} \equiv \frac{1}{16}B_{p-3} \pmod{p}.$$

Proof.

$$\sum_{\substack{0<i<j<p\\ i \text{ even, } j \text{ even}}}\frac{1}{i^2j} = \frac{1}{2^3}\sum_{0<m<n<\frac{p}{2}}\frac{1}{m^2n} \equiv \frac{1}{2^3}\sum_{0<m<n<\frac{p}{2}}\frac{m^{p-3}}{n}$$

$$= \frac{1}{2^3}\sum_{n=1}^{\frac{p+1}{2}}\frac{B_{p-2}(n) - B_{p-2}}{(p-2)n} = \frac{1}{2^3}\sum_{n=1}^{\frac{p+1}{2}}\frac{\sum_{k=0}^{p-2}\binom{p-2}{k}B_kn^{p-2-k} - B_{p-2}}{(p-2)n}$$

$$\equiv -\frac{1}{2^4}\sum_{n=1}^{\frac{p+1}{2}}\sum_{k=0}^{p-3}\binom{p-2}{k}B_kn^{p-3-k} \equiv -\frac{1}{2^4}\sum_{k=0}^{p-3}\binom{p-2}{k}B_k\sum_{n=1}^{\frac{p+1}{2}}\frac{1}{n^{k+2}}$$

$$\equiv -\frac{1}{2^4}\sum_{k=0}^{p-3}\binom{p-2}{k}B_k\left[\sum_{n=1}^{\frac{p-1}{2}}\frac{1}{n^{k+2}} + 2^{k+2}\right]$$

$$\equiv -\frac{1}{2^4}\sum_{k=0}^{p-3}\binom{p-2}{k}B_k\sum_{n=1}^{\frac{p-1}{2}}\frac{1}{n^{k+2}} - \frac{1}{2^2}\sum_{k=0}^{p-3}\binom{p-2}{k}B_k2^k.$$

In the first term we use the facts that $B_k = 0$ for all odd $k > 1$ and $\sum_{n=1}^{\frac{p-1}{2}}\frac{1}{n^{k+2}} \equiv 0 \pmod{p}$ when k is even and $k < p-3$, by (4). Hence we are left to examine the terms, $k = 1$ and $k = p-3$. By (5) we get

$$-\frac{1}{2^4}(-2)\frac{-1}{2}\cdot(-2B_{p-3}) + \frac{1}{2^3}B_{p-3}\cdot\frac{-1}{2} \equiv \frac{1}{16}B_{p-3} \pmod{p}.$$

Concerning the last term, we use (3):

$$\sum_{k=0}^{p-2} \binom{p-2}{k} B_k 2^k = 2^{p-2} B_{p-2}(\tfrac{1}{2}) = (2-2^{p-2})B_{p-2} = 0,$$

since $p-2$ is odd and $p > 3$. □

Theorem. *If p is prime and $p > 3$ then*

$$\sum_{\substack{0<i<j<p \\ i \text{ odd, } j \text{ even}}} \frac{1}{ij} \equiv -\frac{7}{48} p B_{p-3} \pmod{p^2}. \tag{7}$$

Proof. By exploiting three kinds of bijections as in [1], we see that

$$\begin{aligned}
3 \sum_{\substack{0<i<j<p \\ i \text{ odd, } j \text{ even}}} \frac{1}{ij} &= \sum_{\substack{0<i<j<p \\ i \text{ odd, } j \text{ even}}} \frac{1}{ij} + \sum_{\substack{0<i<j<p \\ i \text{ odd, } j \text{ even}}} \frac{1}{(j-i)j} + \sum_{\substack{0<i<j<p \\ i \text{ odd, } j \text{ even}}} \frac{1}{i(p+i-j)} \\
&= \sum_{\substack{0<i<j<p \\ i \text{ odd, } j \text{ even}}} \frac{p}{i(j-i)(p+i-j)} \cdot \frac{p-i+j}{p-i+j} \\
&\equiv p \sum_{\substack{0<i<j<p \\ i \text{ odd, } j \text{ even}}} \frac{p-i+j}{(-i)(j-i)(i-j)^2} \equiv p \sum_{\substack{0<i<j<p \\ i \text{ odd, } j \text{ even}}} \frac{1}{(j-i)^2(-i)} \\
&\equiv p \sum_{\substack{0<i<j<p \\ i \text{ odd, } j \text{ even}}} \frac{1}{i^2 j} \pmod{p^2},
\end{aligned}$$

where in the last passage we replaced $j-i$ by i and $p-i$ by j.

Therefore, combining the above with our two previous lemmas gives the final blow:

$$\begin{aligned}
\sum_{\substack{0<i<j<p \\ i \text{ odd, } j \text{ even}}} \frac{1}{ij} &\equiv \frac{p}{3} \sum_{\substack{0<i<j<p \\ i \text{ odd, } j \text{ even}}} \frac{1}{i^2 j} \equiv \frac{p}{3} \left(\sum_{\substack{0<i<j<p \\ j \text{ even}}} \frac{1}{i^2 j} - \sum_{\substack{0<i<j<p \\ i \text{ and } j \text{ even}}} \frac{1}{i^2 j} \right) \\
&\equiv \frac{p}{3} \left(-\frac{3}{8} B_{p-3} - \frac{1}{16} B_{p-3} \right) \equiv -\frac{7}{48} B_{p-3} \pmod{p^2}.
\end{aligned}$$

□

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