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$S_{2n-1}$. *For the expectations of these random variables we have*

$$2\,E(C_{2n}) = E(Z_{2n}) = 2nu_{2n} - 1 \tag{6.1}$$

*and thus*

$$2\,E(C_{2n}) = E(Z_{2n}) \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (2n)^{\frac{1}{2}} \quad \textit{as } n \longrightarrow \infty . \tag{6.2}$$

(These formulas shows that the density of the zeros and of changes of sign decreses at a fast rate.)

*Proof.* Define new random variables by $Y_j = 1$ if $X_j = 0$, and $Y_j = 0$ if $X_j \neq 0$. Then

$$2\,E(C_{2n}) = E(Z_{2n}) = \sum_{j=1}^{2n-1} E(Y_j) = \sum_{r=1}^{n-1} u_{2r} . \tag{6.3}$$

and (6.1) follows by induction.

## 7. Later returns to the origin.

As a further application of the present elementary approach let us prove an important formula half of which has been proved by rather involved analytical methods [4].

THEOREM 5. *Let $f_{k,2n}$ denote the probability that the $k$-th return to the origin takes place at the $2n$-th step (that is, $f_{k,2n}$ is the probability that $S_{2n} = 0$ and exactly $r - 1$ among the $S_j$ with $1 \leq j < 2n$ vanish). Then*

$$f_{k,2n} = z_{k,2n} - z_{k+1,2n} = \frac{2^k}{2^{2n}} \binom{2n-k}{n} \frac{k}{2n-k} . \tag{7.1}$$

*Proof.* It is clear that $f_{1,2n} = f_{2n}$ and that the $f_{k,2n}$ satisfy the recurrence relation (4.9) and hence also (4.11). If we define $f_{0,2n} = 0$ for $n$ 1 and $f_{0,0} = 1$, then (7.1) is true for $k = 0,1$ and therefore for all $k \geq 0$.

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[4] See, for example, *ibid.*, formula (6.15) of Chapter 12.