

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 3 (1957)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE NUMBERS OF ZEROS AND OF CHANGES OF SIGN IN A SYMMETRIC RANDOM WALK
Autor: Feller, William
Kapitel: 7. Later returns to the origin.
DOI: <https://doi.org/10.5169/seals-33746>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 03.04.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

S_{2n-1} . For the expectations of these random variables we have

$$(6.1) \quad 2 E (C_{2n}) = E (Z_{2n}) = 2nu_{2n} - 1$$

and thus

$$(6.2) \quad 2 E (C_{2n}) = E (Z_{2n}) \sim \left(\frac{2}{\pi}\right)^{1/2} (2n)^{1/2} \text{ as } n \rightarrow \infty .$$

(These formulas shows that the density of the zeros and of changes of sign decreases at a fast rate.)

Proof. Define new random variables by $Y_j = 1$ if $X_j = 0$, and $Y_j = 0$ if $X_j \neq 0$. Then

$$(6.3) \quad 2 E (C_{2n}) = E (Z_{2n}) = \sum_{j=1}^{2n-1} E (Y_j) = \sum_{r=1}^{n-1} u_{2r} .$$

and (6.1) follows by induction.

7. LATER RETURNS TO THE ORIGIN.

As a further application of the present elementary approach let us prove an important formula half of which has been proved by rather involved analytical methods ⁴.

THEOREM 5. Let $f_{k,2n}$ denote the probability that the k -th return to the origin takes place at the $2n$ -th step (that is, $f_{k,2n}$ is the probability that $S_{2n} = 0$ and exactly $r - 1$ among the S_j with $1 \leq j < 2n$ vanish). Then

$$(7.1) \quad f_{k,2n} = z_{k,2n} - z_{k+1,2n} = \frac{2^k}{2^{2n}} \binom{2n-k}{n} \frac{k}{2n-k} .$$

Proof. It is clear that $f_{1,2n} = f_{2n}$ and that the $f_{k,2n}$ satisfy the recurrence relation (4.9) and hence also (4.11). If we define $f_{0,2n} = 0$ for $n \geq 1$ and $f_{0,0} = 1$, then (7.1) is true for $k = 0, 1$ and therefore for all $k \geq 0$.

Reçu le 17 mai 1957.

⁴ See, for example, *ibid.*, formula (6.15) of Chapter 12.