COMMISSION INTERNATIONALE DE L'ENSEIGNEMENT MATHÉMATIQUE (C.I.E.M.)

ON THE TEACHING OF GEOMETRY IN SECONDARY SCHOOLS [1)]

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1. Like in other parts of mathematics the *teaching of geometry has to be based upon the fundamental notions*: *set*, *function*, *relation.* Geometry is a very appropriate field where these notions occur almost automatically; it was a wrong standpoint that such a favorable situation was not sufficiently used neither in geometry nor in other (mathematical or non mathematical) fields.

2. *The teaching of geometry has the aim to contribute to the mathematical culture and education* and not only to cultivate pure geometrical ideas, methods and like that. *Drawing and the space-inspection have to be done together with calculations and reasonings.* Therefore we stress *vectors considerations* as a very far-reaching tool and method. Vector considerations link mathematical formalism and direct mathematical insights.

3. Children perceive bodies, sets, situations; afterwards they see that many *bodies have a structure*: several organized parts that are in some mutual relations as well as in some relation with the whole. E.g. looking on a cube they perceive on it faces, edges, vertices. Children have a certain idea of the notion of *place*, *position*, where a body is located. They have

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1) A lecture at Montclair College, July 1959.

an idea of the *size* of a body and have the opportunity to *diminish* this body as well as to *augment* it.

3. 1. One has to start with *descriptions of mutual positions and relations* of individual bodies e.g. by stating what is common, what is distinct on them etc. The environment and various phenomena are very rich in such situations.

3. 2. Afterwards comes next the step when the pupils *produce*, perform such and such situation, describe them, vary them etc.

3. 3. Intuitive and topological items are much easier to grasp and understand than other situations. Therefore *one has to start with intuitive and topological descriptions, situations.* Next step: *qualitative* situations e.g. bigger, smaller, before, behind, many-few, all-no, at least one, round, pointed etc. Next step: more or less *precise situations like: as many as, five, point, particular numbers, greatest, smallest and numerical approximations.*

3. 4. *In particular, extremality considerations are to be favored everywhere.* A fine example of such considerations are the union (the intersection) of given sets as the minimal (maximal) set containing (contained in) each given set.

4. *Fundamental concepts: interval and convexity.*

4. 1. The fundamental geometrical concept is the interval determined by two given points or position A, B: it is the *shortest* way joining these points. As an experimental fact (use of ruler) one has to accept it as well as its *convexity property: For any two points C, D of the interval AB the interval CD lies on AB.*

4. 2. At the very beginning one defines *convex sets* as any set X such that X contains the whole interval, provided it contains the end-points of the interval:

$$A \in X,\ B \in X \Rightarrow AB \subseteq X\ .$$

4. 3. If two intervals have a common interval their union is again an interval.

4. 4. One learns from the experience the possibility of prolongation of any interval, of multiplication and transfer of any interval etc. Using simple tools (like ruler or a rigid body)

one determines whether two intervals are or are not of a same length.

5. *Vectors.*

5. 1. At the very beginning one distinguishes the path from A to B of the *opposite* path from B to A. And one speaks of the *vector* AB, symbolically $\overrightarrow{AB}$, stressing that one is dealing here with two items: *a set and a direction.* The *opposite vector* $\overrightarrow{BA}$ is linked with $\overrightarrow{AB}$ and is denoted too as $-\overrightarrow{AB}$ or $(-1)\overrightarrow{AB}$ meaning simply that they are of the same magnitude but of opposite direction.

5. 2. *Null-vector.* For any point P one defines the *nullvector* $\overrightarrow{PP}$: its length is 0, its direction is undetermined. One postulates the equality of all the nullvectors $\overrightarrow{PP}$; it is designated by $\vec{0}$.

5. 3. One of the fundamental operations consists to admit the *possibility of a transfer of any vector* v *using any given point as its new origin*: the new vector has to have the same length and the same direction as the given vector v. The construction is carried out experimentally on a piece of paper or other " plane " regions using compass, rigid bodies etc.

5. 3. 1. *Central symmetry.* Given a point C, for each point P one has a unique point $C(P)$ such that

$$\overrightarrow{PC} = \overrightarrow{C\,C(P)} \,.$$

The point C is the middle point of the interval $PC(P)$. The point $C(P)$ is the *central image of* P with respect to C as the *center of symmetry.* For any set S one defines $C(S)$ as the set of $C(P)$, P running over S. The symmetry center of S is each point C such that $S = C(S)$. For instance C is a center of symmetry of the set $S \cup C(S)$, for any set S.

5. 3. 2. *Translation.* Translation of a set S for a vector $\vec{v}$ consists to find the set of all points X' such that $\overrightarrow{XX'} = \vec{v}$, X running over S.

5. 4. *Addition of vectors.* The fundamental definition lies in the postulational equality $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ meaning that

the straight moving from A to B and then from B to C yields the same result as the direct straight moving from A to C. *One has to regard the previous equality more as expression for directions than for magnitudes.*

6. *Sphere.* The union of all intervals CX having a common endpoint C and the same length constitutes the full sphere with C as its center and CX as its radius.

7. *Straight line.*

6. 1. *The straight line is defined as the union of all the intervals containing a given interval or two given points.*

6. 2. Children appreciate very much the jump from interval to straight line which is no interval. This jump is " evident " here and prepares them to grasp the limit process, to come afterwards.

6. 3. The generation of the line AB from the vector $\overrightarrow{AB}$ is to be organized in such a way that one sees how the *real numbers occur and are formed during the process:* one combines geometrical-physical process of prolonging an interval with the process of measuring, counting etc.: One constructs the interval AC so that $\overrightarrow{AB} = \overrightarrow{BC}$ and one writes $\overrightarrow{AC} = 2\,\overrightarrow{AB}$; likewise $3\,\overrightarrow{AB} = \overrightarrow{AD}$ etc. $n\overrightarrow{AB}$ for any positive integer > 1; one writes $\overrightarrow{AB} = 1 \,.\, \overrightarrow{AB}$ and $-\,\overrightarrow{AB} = (-1) \,.\, \overrightarrow{AB} = \overrightarrow{BA}$.

In this way one becomes aware of a basic structure, of the fundamental fact how the line is generated by an interval and how the points of a line are connected with real numbers. One learns about the straight line as well as about the number line; one becomes acquainted with the fundamental mapping between real numbers and points of a line.

6. 4. One becomes aware that the straight line AB is the *minimal* over-set of AB carried up in itself by the translation $\overrightarrow{AB}$. This way of generating the line is very instructive.

6. 5. *Oriented straight line.* The choice of a vector $\overrightarrow{AB} \neq \vec{0}$ enables us *to order* or to *give an orientation* of the straight line AB: its point X *precedes* Y provided the vectors $\overrightarrow{XY}$ and $\overrightarrow{AB}$ have the same direction.

6. 6. *Number line. Numerical notations of points of a line.* Beside the usual notations of points by means of characters one learns *the numerical designation of points which is much more convenient and appropriate.* One speaks of the points 5 or (5), (— 5), (0) etc.

6. 7. *Linearity of a line.* One assumes the *linear property or linearity of any line:* if a line l contains 2 points X, Y then it contains also the corresponding line XY.

6. 8. One proves readily then that every line is determined by any two of its points.

PROOF. Let l be a line and A, B points such that l be the union of all intervals containing A, B; we write $l = 1\,(A, B)$. Let $C, D \in 1\,(A, B)$. Then

$$1\,(C, D) \subseteq 1\,(A, B)\,, \qquad (*)$$

the line being a linear set. Now, the points A, B are obtainable starting from CD and prolonging CD; hence, $A, B \in 1\,(CD)$ and the linearity property implies $1\,(A, B) \subseteq 1\,(C, D)$. This relation together with the relation (*) implies that $1\,(AB) = 1\,(CD)$. Q.E.D.

6. 9. *Linearity of a set. A set X* possesses the *linear property* if with every two distinct points it contains the corresponding straight line.

The linearity property is the analogue of the convexity property. Every linear set is convex; the converse needs not hold.

6. 10. It is inadmissible that one teaches everywhere about straight line first and that an interval *is defined* as a part of straight line.

7. *Triangle, plane.*

7. 1. The set consisting of 3 points A, B, C is not convex. To produce a convex set containing A, B, C one has to adjoin first of all the intervals AB, BC, CA; one gets in this way the closed path $AB \cup BC \cup CA$. If this path is not an interval, it is non convex. In order to make it convex, one adjoins the points of the intervals AX where $X \in BC$, as well as the points of the intervals BY where $Y \in BC$ and the points CZ where $Z \in CA$. It is not evident at all that the union of all these points is convex. *One postulates its convexity.*

7. 2. Definition. If A, B, C are 3 given points the corresponding triangle $\triangle ABC$ is the *minimal convex overset.* The triangle $\triangle ABC$ is bounded by the edges AB, BC, CA and their union $AB \cup BC \cup CA$. One assumes that this union is not an interval; if this union is an interval, one speaks of a degenerated triangle.

7. 3. Here again one does not forget the fundamental postulational equalities: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, $\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$, $\overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{CB}$.

7. 4. Any triangle $\triangle$ contains various subsets and is contained in various sets.

7, 4. 1. In particular, there are triangles contained in $\triangle$ and containing $\triangle$ respectively. The union of all triangles containing $\triangle$ is not a triangle; this union is called the plane. One could define the planes in this way.

7. 4. 2. There are very various " *plane* " *extensions* of a triangle, e.g. taking the union of all the rays AX, where $X \in BC$. Just this extension is very important.

Here is a particular extension: let A_0 be the middle point of BC; for any point $P \in BA \cup AC$ let P_a be the point such that $\overrightarrow{PA_0} = A_0 P_a$; then the union of all the intervals PP_a is a well determined set: (cf. § 9. 5).

7. 4. 3. *Definition.* The *plane* defined by 3 points is the *minimal linear* set containing these points.

Consequently, *one postulates the linearity of every plane*: if μ is any plane and if X, Y are 2 distinct points of μ then $1(X, Y) \subseteq \mu$.

7. 5. *Interior of a triangle.* The *interior* of a triangle is obtained from this triangle by dropping the border line $AB \cup BC \cup CA$. The *exterior* of the triangle is obtained from the plane ABC by dropping the closed triangle ABC.

8. *Circle.*

8. 1. A *circle* is defined as common part of a plane and a sphere. One supposes that a circle has more than one point. Any point is considered as a degenerated circle.

8. 2. One learns that any triangle contains (is contained) a maximal (in a *minimal*) circle.

9. *Parallelogram.*

9. 1. Parallelogram is any convex quadrangle which is centrally symmetrical.

9. 2. Fundamental construction of a parallelogram consists of two intervals which bisect one another.

9. 3. Its fundamental postulational property consists in the fact that the sum of its edges issuing in a vertex equals the diagonal issuing in the same vertex.

9. 4. The role of a parallelogram is to yield *parallel lines*: opposite edges are parallel as well as their linear bearers.

9. 5. *Parallelogram and translation.* In each parallelogram $ABCD$ the translation $\overrightarrow{CD}$ mapps AB onto DC. Opposite sides are equal.

9. 6. For vectors one has $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (*commutative law* for addition of vectors).

10. A *tetrahedron* is defined in a similar manner as a triangle: *it is the least convex set containing 4 given points.* Usually one supposes that these points—vertices—are not located in a plane.

It is very instructive to become aware how the tetrahedron $ABCD$ is generated: First of all one has the edges AB, AC, AD; BC, BD, CD. Then one has 4 faces: ABC, ABD, ACD, BCD; finally one has the union of edges AX where $X \in BCD$ etc. Consequently, in connection with a four-point-set $\{A, B, C, D\}$ we have in particular the following sets:

Edges, 6 in number,
triangles, 4 in number,
the tetrahedron as the minimal *convex* overset,
the space as the minimal *linear* overset.

minimal oversphere: sphere the border of which contains the points A, B, C, D.

11. *Angle. Rotation in a plane around a point.*

11. 1. In practice one is dealing not only with point-spans or distances but also with direction-spans or angles.

Simple tools (ruler and goniometer) enable us to measure them.

Primarily, angles are determined by directions, rays, vectors, etc. There are various definitions of an angle. We shall indicate one based on set considerations.

11. 2. Definition. Let Oa, Ob be a given pair of coinitial rays that are no located on a line; the *minimal* convex set containing the rays Oa, Ob is called the convex (and closed) angle Oa, Ob and is denoted by

$$\sphericalangle (Oa, Ob) \quad \text{or} \quad \sphericalangle (Ob, Oa) \quad \text{or} \quad \sphericalangle (aOb) \,. \tag{1}$$

O is called the *vertex*; Oa, Ob are the *sides* of the angle (1). The *open angle* corresponding to (1) is obtained from (1) by removing the sides.

11. 2. 1. We see that (1) is the convex part of the plane containing Oa, Ob and bordered by $Oa \cup Ob$; the other part of the plane is called the *associated open angle* of (1); the *closed associated* of (1) is the union of the open associated and of Oa, Ob.

11. 2. 2. If Oa, Ob exhaust a line l, then every plane p containing the line l is the union of two open half planes and of l; each of these open half planes is the open angle Oa, Ob; O is the vertex. The closed angle Oa, Ob is the union of the open angle Oa, Ob and of the sides Oa, Ob.

11. 2. 3. The convex angle Oa, Oa is just the *ray* Oa; the associated open angle is any plane $p \supset Oa$ from which Oa is removed.

11. 2. 4. *Oriented angles.* The oriented angle Oa, Ob, symbolically $\overset{\curvearrowleft}{\sphericalangle} (Oa, Ob)$ means the angle Oa, Ob and that Ob is the *first* and Ob is the *second* side of the angle.

11. 2. 5. If $\sphericalangle (Oa, Ob)$ is a given angle an dif x is any ray or vector which is $\parallel Oa$, one defines also the angle $\sphericalangle (x, Ob)$. Analogously, one defines $\sphericalangle (x, y)$, for every ray or vector y such that $y \parallel Ob$.

11. 2. 6. In a similar way one defines the *oriented angle* $\overset{\curvearrowleft}{\sphericalangle} (x, y)$.

11. 3. *Equality of angles.* Definition. *Two angles are equal if there exists an isometric transformation of one angle on to the other angle.*

11. 3. 2. Two oriented angles are equal if there exists an isometric mapping of one angle onto the other angle carrying the first side and the second side of one angle onto the first side and the second side of the other angle respectively.

11. 3. 3. For any ordered pair (x, y) of vectors or rays *one postulates the equality of the oriented angles* $\overset{\nwarrow}{\sphericalangle}(x, y)$, $\overset{\nwarrow}{\sphericalangle}(x', y')$, where $x' \parallel x$, $y' \parallel y$.

11. 4. *Rotation.* Let α be any oriented angle; let p be a plane and O a point of p; then the rotation of p around O for α is the isometrical transformation

$$\mathrm{P} \rightarrow \alpha P$$

of the plane p onto itself such that $\alpha O = O$ and that for every other point P of p one has

$$\overset{\nwarrow}{\sphericalangle}(PO\,\alpha\,(P)) = \alpha\,.$$

11. 5. The *number circle-line* is a determined uniform mapping of the real numbers on a circle-line: on the circle line one determines an oriented arc equal to the radius and used as unit arc; its endpoints are designated by 0 and 1 in such a way that the interior of the circle is at the left when one goes from 0 to 1 the shorter way. The real numbers are used as in the case of number-line. The difference is that to every number corresponds a unique point of the circle-line but conversely to every point corresponds a set of numbers whose difference is a multiple of 2π. In particular, to numbers $0, 2\pi, \ldots, 2k\pi, \ldots$ $(k = 0, \pm 1, \pm 2, \ldots)$ is associated the same point of the circle-line.

11. 5. 1. The locating of real numbers on the circle-line is a fundamental transformation, *say* k; its inverse, say ^{-}k, associates to every point P of the circle line every number ^{-}kP which could be equivalently used as measure for the angle $\overset{\nwarrow}{\sphericalangle}(\overrightarrow{OX}, \overrightarrow{OP})$, where $\overrightarrow{OX}$ is the radius vector corresponding to the first point of the unit-arc.

The multivalency of measures of angles is one of the most intricate items in mathematics.

12. *Coordinate plane. Plane and its cartesian coordinate systems.*

12. 1. Given any ordered pair of non collinear radius vectors $\overrightarrow{OE_1}$, $\overrightarrow{OE_2}$, for every ordered pair (x, y) of real numbers one has the vectors $x\overrightarrow{OE_1}$, $y\overrightarrow{OE_2}$ and their sum

$$x\overrightarrow{OE_1} + y\overrightarrow{OE_2} . \tag{1}$$

The point (x, y) whose radius vector equals (1) is well-determined; all these points form the plane which is determined by the points O, E_1, E_2. Any point P of this plane has a unique analytic or numerical notation (x, y), where x, y are real numbers satisfying

$$\overrightarrow{OP} = x\overrightarrow{OE_1} + y\overrightarrow{OE_2} .$$

12. 2. The simplest case is the one where $OE_1 \perp OE$ and $|OE_1| = |OE_2| = 1$.

One speaks then of an *orthogonal cartesian coordinate system.*

12. 3. Analogously, if p is a plane and $\vec{r}_1$, $\vec{r}_2$ are any two orthogonal vectors of the plane of the same length, then by chosing a point O of the plane for the pole, one gets the unique representation of every point P of the plane just as ordered pair (x, y) of real numbers x, y satisfying

$$\overrightarrow{OP} = x\vec{r}_1 + y\vec{r}_2 .$$

The axes of the " coordinate system " $(\vec{r}_1, \vec{r}_2)$ are the carriers of the vector $\vec{r}_1$ and $\vec{r}_2$ respectively; they are two well-determined number lines.

Consequently, *in a plane a coordinate system is given in the first place by an ordered pair of independent* (non parallel) *vectors and not by two infinite straight lines.*

12. 4. *Coordinate space.*

In a completely analogous way one gets the analytical numerical representation of points P of the space by chosing an ordered triple of independent radii-vectors or any ordered triple of independent vectors $\vec{e}_0$, $\vec{e}_1$, $\vec{e}_2$ and a point O as pole: where

$$\overrightarrow{OP} = x\vec{e}_0 + y\vec{e}_1 + z\vec{e}_2 .$$

The simplest case is the one where $\vec{e}_0$, $\vec{e}_1$, $\vec{e}_2$ are orthonormal: each of length 1, each orthogonal to any other.

13. *Cosine and sine.* For any point P lying on the number circle-line one has the angle $\sphericalangle XOP$ and its measures ^{-}kP. On the other hand one has the coordinate system OX, OY, in which the point P has a definite representation: the abscissa of P is called the cosine of the angle xOP or of each number ^{-}kP, the ordinate of P is called the sine of the angle XOP or of every number ^{-}kP and one writes $P = (\cos {}^{-}kP, \sin {}^{-}kP)$.

14. *Orthogonal projections on lines, planes, vectors. Scalar multiplication of vectors.*

14. 1. Let L be a line or a plane and P a point. The line $L\,(P)$ which contains the point P and is orthogonal to L is well determined. The point P' in which $L\,(P)$ meets the given L is well determined and is called the *projection of P on L.*

14. 2. The projection of a *set* S on L is defined as the set of projections of points of S:

$$proj\ S = \left\{ proj\ X \,\middle|\, X \in S \right\}.$$

14. 3. Let L be an interval or a non null-vector or a plane piece; then the minimal linear set lL containing L is well determined; one defines the projection on L as the projection on lL.

14. 4. It is to be noticed that one distinguishes two kinds of projecting a vector $\overrightarrow{AB}$ on a vector $\overrightarrow{CD}$: the *vector projection* of $\overrightarrow{AB}$ on $\overrightarrow{CD}$ is the *vector* $\overrightarrow{A'\,B'}$ where A' and B' are projections of A and B on CD; the *scalar projection* of the vector AB on the vector $\overrightarrow{CD}$ is the number $|\,AB\,|\ .\ \cos \sphericalangle\,(\overrightarrow{CD}, \overrightarrow{AB})$.

Analogously, the vector projection and scalar projection of a vector $\overrightarrow{AB}$ on an oriented line l is the vector $\overrightarrow{A'\,B'}$ and the number $|\,AB\,|\,\cos \sphericalangle\,(l, \overrightarrow{AB})$ respectively; here A', B' denote the projections on l of A and B.

The scalar projection of vectors is a very important operation.

14. 5. The two kinds of projections of vectors are distributive with respect to the addition of vectors.

14. 6. *Scalar multiplication of vectors.* The scalar product $\vec{a}\vec{b}$ of the vectors $\vec{a}$, $\vec{b}$ is the product of their magnitudes and the cosine of their angle:

$$\vec{a}\vec{b} = ab \cos \sphericalangle (\vec{a}, \vec{b}) .$$

One sees that $\vec{a}\vec{b} = a\, b_{\vec{a}} = a_{\vec{b}} \,.\, b$.

In particular, the cosine of the angle between two unit vectors equals their scalar product.

If $\vec{a} \perp \vec{b}$ then $\vec{a}\vec{b} = 0$ and vice versa: if $\vec{a}\vec{b} = 0$ then $\vec{a} \perp \vec{b}$.

15. *Isometries in space: translations, rotations, symmetries* (with respect to a point, line, plane). They are defined in usual manner.

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(A suivre)