

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 9 (1963)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON IMPLICIT FUNCTION THEOREMS AND THE EXISTENCE OF SOLUTIONS OF NON-LINEAR EQUATIONS

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Kapitel: 5. An example.

DOI: <https://doi.org/10.5169/seals-38780>

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is solvable is open with respect to the interval $[0, 1]$. This follows from $a)$. It is also closed, for if $\tilde{\lambda}$ is the supremum of A then there exists a point $\lambda^* \in A$ with $|\tilde{\lambda}^* - \lambda| \|\omega_0 - \omega_1\| < c$. Thus it follows from $a)$, if ω_0 is replaced by $\omega_0 - \lambda^*(\omega_0 - \omega_1)$, that $\tilde{\lambda} \in A$. Hence $A = [0, 1]$ and (4.1) has a solution for all $\omega \in B_2$.

Proof of Theorem 4.1 a. Let $\omega_1 \in B_2$ and $u_0 \in D$ with $Tu_0 = \omega_0$ be given. Then the points $\omega = \omega_0 + \lambda(\omega_1 - \omega_0)$, $0 \leq \lambda \leq 1$, are bounded:

$$\|w\| \leq \max(\|\omega_0\|, \|\omega_1\|) = A.$$

Because of $\gamma')$ there exists a number R with $\|Tu\| > A$ for all u in the set $\{u \in D: \|u\| \geq R\}$.¹⁾

Then the same conclusion as in the proof of Theorem 4.1 with $c = c(R)$ applied to $\|u\| \leq R$ shows that $Tu = \omega_1$ is solvable by an element u_1 with $\|u_1\| < R$ for which the assumptions of Theorem 4.1 with $c = c(R)$ hold. This implies the existence of a sphere $\|\omega - \omega_1\| < c$ with the asserted properties.

5. AN EXAMPLE.

The simple example $Tu = \tan u$, given only for illustration purposes, shows that Theorem 4.1 is general enough to cover cases in which either the domain D is not the whole space B_1 or $Tu = \omega$ does not have a unique solution, although this equation is solvable for all $\omega \in B_2$.

Let $B_1 = B_2 = B$ be the Banach space of real numbers. Then by Theorem 4.1 the equation

$$Tu \equiv \tan u = w, \quad u, w \in B,$$

is solvable for all $\omega \in B$.²⁾

Proof. We choose

$$Kv = \frac{v}{\cos^2 u} \quad \text{for} \quad u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

¹⁾ This set may be empty.

²⁾ This is not true for complex numbers as $\tan u = i$ is not solvable.

Then by the mean value theorem and because

$$\frac{d}{du} \frac{1}{\cos^2 u} = \frac{2 \sin u}{\cos^3 u},$$

is increasing for increasing

$$u \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right),$$

it follows that

$$m(u) = \frac{1}{\cos^2(u+r)} - \frac{1}{\cos^2 u} \quad \text{for} \quad 0 \leq u < \frac{\pi}{2} \quad \text{and} \quad u+r < \frac{\pi}{2}.$$

In the following we restrict ourselves to these u .

From the above we get

$$(\|K^{-1}\|^{-1} - m)r > \left(\frac{1}{\cos^2(u+r)} - \frac{4r}{\cos^3(u+r)} \right) r, \quad 0 < r < \frac{\pi}{2} - u.$$

Now choosing r as the smallest positive solution of $r = r(u) = \frac{1}{8} \cos(u+r)$, which implies $u+r < \frac{\pi}{2}$, we get

$$(\|K^{-1}\|^{-1} - m)r > \frac{1}{16 \cos(u+r)} > \frac{1}{16}. \quad ^1)$$

The same is true for $-\frac{\pi}{2} < u < 0$ as can be proved in the same way. Thus the conditions of Theorem 4.1 are valid. In particular $\gamma)$ is true for $c = \frac{1}{16}$.

6. INVERSE FUNCTION THEOREMS (continued).

As was indicated by the example $\tan u = \omega$ in the last chapter, the assumptions of the Theorems 4.1 and 4.1 a are not sufficient to insure that the operator T will have an inverse

¹⁾ Here we use the fact that u is real.