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is solvable is open with respect to the interval [0, 1]. This follows from *a)*. It is also closed, for if $\tilde{\lambda}$ is the supremum of $\Lambda$ then there exists a point $\lambda^* \in \Lambda$ with $|\tilde{\lambda}^* - \lambda| \, \| w_0 - w_1 \| < c$. Thus it follows from *a)*, if $w_0$ is replaced by $w_0 - \lambda^* (w_0 - w_1)$, that $\tilde{\lambda} \in \Lambda$. Hence $\Lambda = [0, 1]$ and (4.1) has a solution for all $w \in B_2$.

*Proof of Theorem 4.1* a. Let $w_1 \in B_2$ and $u_0 \in D$ with $Tu_0 = w_0$ be given. Then the points $w = w_0 + \lambda (w_1 - w_0)$, $0 \leqq \lambda \leqq 1$, are bounded:

$$\| w \| \leqq \max (\| w_0 \|, \| w_1 \|) = A .$$

Because of $\gamma'$) there exists a number $R$ with $\| Tu \| > A$ for all $u$ in the set $\{ u \in D : \| u \| \geqq R \}$.[1])

Then the same conclusion as in the proof of Theorem 4.1 with $c = c(R)$ applied to $\| u \| \leqq R$ shows that $Tu = w_1$ is solvable by an element $u_1$ with $\| u_1 \| < R$ for which the assumptions of Theorem 4.1 with $c = c(R)$ hold. This implies the existence of a sphere $\| w - w_1 \| < c$ with the asserted properties.

## 5. An example.

The simple example $Tu = \tan u$, given only for illustration purposes, shows that Theorem 4.1 is general enough to cover cases in which either the domain $D$ is not the whole space $B_1$ or $Tu = w$ does not have a unique solution, although this equation is solvable for all $w \in B_2$.

Let $B_1 = B_2 = B$ be the Banach space of real numbers. Then by Theorem 4.1 the equation

$$Tu \equiv \tan u = w , \quad u, w \in B ,$$

is solvable for all $w \in B$.[2])

*Proof.* We choose

$$Kv = \frac{v}{\cos^2 u} \quad \text{for} \quad u \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right).$$

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1) This set may be empty.
2) This is not true for complex numbers as tan u = i is not solvable.

Then by the mean value theorem and because

$$\frac{d}{du}\frac{1}{\cos^2 u} = \frac{2 \sin u}{\cos^3 u},$$

is increasing for increasing

$$u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

it follows that

$$m(u) = \frac{1}{\cos^2(u+r)} - \frac{1}{\cos^2 u} \quad \text{for} \quad 0 \leqq u < \frac{\pi}{2} \quad \text{and} \quad u+r < \frac{\pi}{2}.$$

In the following we restrict ourselves to these $u$.

From the above we get

$$(\| K^{-1} \|^{-1} - m)r > \left(\frac{1}{\cos^2(u+r)} - \frac{4r}{\cos^3(u+r)}\right)r, \quad 0 < r < \frac{\pi}{2} - u.$$

Now choosing $r$ as the smallest positive solution of $r = r(u) = \frac{1}{8}\cos(u+r)$, which implies $u+r < \frac{\pi}{2}$, we get

$$(\| K^{-1} \|^{-1} - m)r > \frac{1}{16 \cos(u+r)} > \frac{1}{16}.$$ [1])

The same is true for $-\frac{\pi}{2} < u < 0$ as can be proved in the same way. Thus the conditions of Theorem 4.1 are valid. In particular $\gamma$) is true for $c = \frac{1}{16}$.

## 6. Inverse function theorems (continued).

As was indicated by the example $\tan u = w$ in the last chapter, the assumptions of the Theorems 4.1 and 4.1 $a$ are not sufficient to insure that the operator $T$ will have an inverse

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1) Here we use the fact that u is real.