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Artikel: ON IMPLICIT FUNCTION THEOREMS AND THE EXISTENCE OF SOLUTIONS OF NON-LINEAR EQUATIONS

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Corollary 9.2. The equation (9.9) with T satisfying the assumptions of Theorem 9.1 has at least one solution if for at least one $u_0 \in D$, with $\varphi(\lambda)$ the same as in the proof of Theorem 9.1, and

$$w(\lambda) = w_0 + \lambda(w_1 - w_0), \tag{9.10}$$

the operators

$$(T'_{(\varphi(\lambda))})^{-1} = (T^{-1})'_{(w(\lambda))}, \quad 0 \leqq \lambda < 1,$$

exist and are bounded uniformly in λ, or equivalently, if $T'^{-1}_{(u_0)}$ exists as a bounded operator and

$$\| (T'_{(\varphi(\lambda))})^{-1} \| = \| (T^{-1})'_{(w(\lambda))} \|,$$

remains finite with increasing λ from 0 to 1.

Example. It is well known that the equation

$$Tz \equiv \tan z = w, \; z, \; w \text{ complex numbers},$$

is not solvable only for $w = \pm i$. Theorem 9.1 immediately shows that the equation is solvable for all $w \neq \pm i$. For

$$(T^{-1})'_{(w)} = \frac{1}{1+w^2},$$

and, with $w_{0_1} = 0 = \tan 0$ and $w_{0_2} = 1 = \tan \frac{\pi}{4}$, all points of the complex number plane can be reached on straight lines (9.10) from either 0 or 1 such that $\frac{1}{1+(w(\lambda))^2}$ remains bounded with the only exceptions $w = \pm i$.

10. Completely continuous operators, neighborhood and inverse function theorems.

The assumptions of the theorems can be partially weakened if the non-linear equation can be written in the form

$$u = Vu, \tag{10.1}$$

with a completely continuous operator V. Complete continuity is the most used and most convenient aid for stating existence theorems. Therefore, very may existence theorems use it in their proofs and subtle investigations have been made to show that special operators have this property.[1] Two main ways for using the complete continuity should be emphasized: The fixed point principle based on the Schauder-Tychonoff fixed point theorem[2] and the Leray-Schauder method[3] which is a generalization of the theory of degree of a mapping due to Brouwer. One of the main and nicest results which is important for the applications is the following alternative [4] as basis for a priori estimates:

Theorem 10.1. If V is defined on a Banach space B with range in B and if V is completely continuous then either (10.1) has a solution or the set $U = \{ u \colon u = \lambda V u,\ 0 < \lambda < 1 \}$ is not bounded.

But the boundedness of the set U is, of course, only a sufficient condition and in many cases Theorem 10.1 is not applicable. Moreover, the conditions do not imply the existence of a solution in the neighborhood of a given solution. Therefore, the following theorems, which are analogous to some of the above theorems, may be useful.

As the Fréchet derivative of a completely continuous operator is also completely continuous [5] it is no great restriction of generality if we assume that the linear approximation K of $I-V$, which occurs in Theorems 3.1 and 4.1, has the form $I-L$ with a linear completely continuous operator L. Since a completely continuous operator has only a point spectrum [6], $(I-L)^{-1}$ exists as a bounded linear operator defined on the whole Banach space B_2 if and only if L does not have the eigenvalue [7] 1. Therefore, from Theorem 3.1 there follows immediately the

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1) See, for example, M. A. Krasmosel'skii [15].
2) See section 2f.
3) J. Leray et J. Schauder [16].
4) H. Schaefer [17] grave an elegant proof for this theorem in a more general form.
5) If T is differentiable at u and completely continuous the operators $A(c)$: $A(c)k = \dfrac{T(u+ck)\text{-}Tu}{c}$ with real c > 0 are also completely continuous and $\| A(c)k\text{-}T'_{(u)}k \| = \| k \| \varphi(\| ck \|)$, $\varphi(\delta) \to 0$ as $\delta \to 0$. This implies complete continuity of $T'_{(u)}$. See. for example, A. N. Kolmogorov and S. V. Fomin [18] I, p. 114.
6) See, for example, A. N. Kolmogorov and S. V. Fomin [18] I, p. 117 and 120,
7) λ is an eigenvalue of L if $Lu = \lambda\, u$ has a non-trivial solution.

Neighborhood Theorem 10.2. Let the equation (10.1) have the solution u_0 and let there exist a completely continuous linear operator L, which does not have the eigenvalue 1, and a number $m < \| (I-L)^{-1} \|^{-1}$ such that

$$\| (V-L)v-(V-L)u \| \leqq m \| v-u \| \quad \text{for} \quad u, v \in S(u_0, r), (r > 0).$$

Then an $\Omega = (u_0, r_0, a, b)$-neighborhood of $\tilde{T} = I - V$ exists for which the equation

$$Tu = \theta, \quad T \in \Omega, \quad u \in S(u_0, r_0), \tag{10.2}$$

is uniquely solvable. The solution $u(T)$ is continuous at $T = \tilde{T}$, i.e.

$$\| u(T) - u_0 \| \to 0 \quad \text{as} \quad \| Tu_0 \| \to 0 .$$

For the special case $Tu = \tilde{T}u - w$ this theorem shows the existence of a local inverse of $\tilde{T}$.

Inverse Function Theorem 10.2 a. If $\tilde{T}u_0 = u_0 - Vu_0 = w_0$ and if the other assumptions of Theorem 10.1 are satisfied then $\tilde{T} = I - V$ has a local inverse, i.e. there exist positive numbers r and b such that

$$u = Vu + w, \quad \| w - w_0 \| < b, \quad \| u - u_0 \| < r,$$

has a unique solution $u(w)$. Moreover $u(w)$ is continuous at w_0.

These theorems mean, in other words, that the existence of a local neighborhood of $\tilde{T} = I - V$ and u_0 in which the equation (10.1) is uniquely solvable or the existence of a local inverse $\tilde{T}^{-1}$ can only fail if the corresponding linear equation $u = Lu$ is not uniquely solvable.

The above theorems are local theorems insuring the existence of a solution in the neighborhood of a given solution. We now state a global inverse function theorem for the equation

$$u = Vu + w, \tag{10.3}$$

with completely continuous V:

Theorem 10.3 *a)* Let $T = I - V$ with a completely continuous operator V be defined for all $u \in B_1$.

b) For each $u_0 \in B_1$ let there exist a linear operator $L = L_0$ with bounded operator $(I-L)^{-1}$, defined in a neighborhood of u_0, and a number $m = m_0 < \| (I-L)^{-1} \|^{-1}$ such that

$$\| (V-L)u - (V-L)v \| \leqq m \| u - v \| \quad \text{for} \quad u, v \in S(u_0, r), \ r > 0 .$$

c) Let the sets

$$U(g) = \{u : u = Vu + w, \ w \in g\},$$

for each straight line

$$g = w_0 + \lambda (w_1 - w_0), \ 0 \leqq \lambda \leqq 1, \ w_0, \ w_1 \in B_1,$$

be bounded:

$$\| U(g) \| \leqq C(g).$$

Then the equation (10.3) has a solution $u = u(w)$ for all $w \in B_1$ and each point $(w, u(w))$ has a (u, r, a, b)-neighborhood.

This theorem is related to Theorem 10.1 concerning the fact that the condition c represents an a priori estimate. However, it is easy to show that the conditions a and c alone are not sufficient for the existence of a solution for each $w \in B_1$.

α) Condition b is satisfied if V has a derivative $V'_{(u)}$ for all $u \in B_1$ and $(I - V'_{(u)})^{-1}$ exists as a bounded operator. This holds true if $V'_{(u)}$ does not have the eigenvalue 1 since $V'_{(u)}$ is completely continuous.

β) Condition c is satisfied if there exists an a priori estimate for the equation (10.3) of the form

$$\| u \| \leqq C \| w \|,$$

or if the condition

$$\| Tu \| = \| (I - V) u \| \to \infty \quad \text{as} \quad \| u \| \to \infty,$$

holds. Therefore, this theorem can be regarded as a certain " generalization " of Theorem 4.1 *a* for completely continuous V. As a matter of fact, the proof is quite analogous.

Proof of Theorem 10.3. α. Let $u_0 \in B_1$ and $Tu_0 = (I - V) u_0 = w_0$. Then from Theorem 3.1 with $K = I - L$ it follows that an open neighborhood of w_0, $\| w - w_0 \| < a$, exists such that (10.3) is solvable for these w.

β. Let $\tilde{w}$ be an arbitrary point of B_1 and let u_0, w_0 be as above. Then the set Λ of all λ, for which

$$Tu = \lambda \tilde{w} + (1 - \lambda) w_0 , \quad 0 \leqq \lambda \leqq 1 ,$$

is solvable, is non-void and open with respect to [0, 1] according to α.

λ. We show that Λ is also closed. Let $\lambda_n \in \Lambda$, $n = 1, 2, \ldots$, be a sequence which converges to λ^*. According to condition c the solutions u_n of $u = Vu + w_n$, $w_n = \lambda_n \tilde{w} + (1 - \lambda_n) w_0$, are bounded. Because of the complete continuity of V there exists a subsequence u_{n_i} such that Vu_{n_i} converges to some element s of the Banach space B_1.

Let $w^* = \lambda^* \tilde{w} + (1 - \lambda^*) w_0$. Then the sequence u_{n_i} converges to $u^* = s + w^*$ in norm. The element u^* is a solution of the equation $u = Vu + w^*$ since

$$\| u_{n_i} - Vu_{n_i} - w_{n_i} \| = 0 \quad \text{for} \quad i = 1, 2, 3, \ldots ,$$

and because of the continuity of the norm. Hence $\lambda^* \in \Lambda$ and, therefore, $\Lambda = [0, 1]$.

11. Completely continuous operators, global existence theorems using the Schauder fixed point theorem.

The previous theorems, even the global ones, are derived, roughly speaking, by applying neighborhood theorems and exhausting a domain on the boundary of which the assumptions fail to hold. Here the question suggets itself whether or not corresponding conditions in a shell near the boundary suffice for existence. This indeed is possible for equations

$$u = Vu , \tag{11.1}$$