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VERTEX POINTS OF FUNCTIONS

by Ali R. AMIR-MOÉZ

For f a real function of n variables, usually the Hessian matrix is studied in connection with Gaussian and mean curvatures of $f(x_1, \dots, x_n)$. In this paper we study other properties of f in a neighborhood of a point. In particular we get a method for obtaining vertex points of the function f . We also generalize the idea to some complex cases.

1. DEFINITIONS AND NOTATIONS

Let f a function of complex variables x_1, \dots, x_n be of class C'' in x_1, \dots, x_n , and $\bar{x}_1, \dots, \bar{x}_n$, in a neighborhood of a point. Then f is called unitarily analytic if

$$\frac{\partial^2 f}{\partial x_i \partial \bar{x}_j} = \overline{\left(\frac{\partial^2 f}{\partial \bar{x}_i \partial x_j} \right)}.$$

Theorem: Let f be of class C'' in $x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n$ in a neighborhood of a point, and

$$\frac{\partial f}{\partial \bar{x}_k} = \overline{\left(\frac{\partial f}{\partial x_k} \right)}.$$

Then f is unitarily analytic.

The proof is quite simple and we omit it. Note that the converse is not necessarily true.

2. TANGENT QUADRIC

Let f be unitarily analytic in a neighborhood of (c_1, \dots, c_n) .

Let, for example, $\frac{\partial f}{\partial c_1}$ be the value of $\frac{\partial f}{\partial x_1}$ at (c_1, \dots, c_n) , and

$f_c = f(c_1, \dots, c_n)$. Then

$$(x_1 - c_1 \dots x_n - c_n) \begin{bmatrix} \frac{\partial^2 f}{\partial c_1 \partial \bar{c}_1} & \dots & \frac{\partial^2 f}{\partial c_1 \partial \bar{c}_n} \left(\frac{\partial f}{\partial c_1} \right) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial^2 f}{\partial c_n \partial \bar{c}_1} & & \frac{\partial^2 f}{\partial c_n \partial \bar{c}_n} \left(\frac{\partial f}{\partial c_n} \right) \\ \frac{\partial f}{\partial c_1} & & \frac{\partial f}{\partial c_n} & f_c \end{bmatrix} \begin{bmatrix} x_1 - c_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n - c_n \\ 1 \end{bmatrix} = 0 \quad (2.1)$$

is called the tangent quadric of f at (c_1, \dots, c_n) . We shall study only the cases that at least one of the first or second derivatives is not zero. It is clear that the tangent plane of (2.1) at (c_1, \dots, c_n) is the same as the tangent plane of $f = 0$ at this point.

Let the matrix of (2.1) be A , $\xi = (x_1 - c_1 \dots x_n - c_n)$, and $\eta = (0 \dots 0 \ 1)$. Then by section 8 of [1]

$$\xi A \eta^* = 0 \quad (2.2)$$

is the tangent plane of (2.1) at (c_1, \dots, c_n) . Here η^* is the conjugate transpose of η .

We easily see that (2.2) can be written as

$$\sum_{i=1}^n \frac{\partial f}{\partial c_i} (x_i - c_i) = 0. \quad (2.3)$$

3. MATRICES RELATED TO f

Besides A there are other matrices of some interest. We denote the matrix of the quadratic form of (2.1) by Q . The projection on the normal and tangent plane are of some interest. We denote the projection on the normal by P , and clearly $I - P$ is the projection on the tangent plane where I is the identity matrix. It is easy to see that $P = (P_{ij})$, where

$$P_{ij} = \frac{\left(\frac{\partial f}{\partial x_i}\right) \frac{\partial f}{\partial x_j}}{\sum \left|\frac{\partial f}{\partial x_i}\right|^2}.$$

This is proved by considering the inner product of a vector $\xi = (x_1, \dots, x_n)$ and a unit vector on

$$\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right).$$

3. QUADRIC CURVATURE

If (2.1) becomes of the form

$$[\sum a_i (x_i - c_i)] [\sum \bar{a}_i (\bar{x}_i - \bar{c}_i)] = 0,$$

then $f(x_1, \dots, x_n)$ is called doubly flat at (c_1, \dots, c_n) . Suppose (2.1) does not have this form. Then by sec 6 of [1] centers (x_1, \dots, x_n) of (2.1) may be obtained by

$$\xi Q = - \left(\frac{\partial f}{\partial \xi}\right), \tag{3.1}$$

where the row matrix ξ is:

$$\xi = (x_1 - c_1 \dots x_n - c_n), \text{ and } \frac{\partial f}{\partial \xi} = \left(\frac{\partial f}{\partial c_1} \dots \frac{\partial f}{\partial c_n}\right).$$

The equation (3.1) is a system of n linear equations in n unknowns.

The following cases may occur:

I. Let Q be non-singular. Then the quadric has a unique center which is called the center of quadric curvature of $f(x_1, \dots, x_n)$ at $\gamma = (c_1, \dots, c_n)$. Let

$$\xi = - \left(\frac{\partial f}{\partial \xi}\right) Q^{-1}.$$

Then the center is the point defined by $\xi - \gamma$.

II. Let the rank of Q be k , and centers exist. Then these centers are solutions of

$$\xi_k = \xi E = - \left(\frac{\partial f}{\partial \xi} \right) EQ^{-1}, \quad (3.2)$$

where Q^{-1} is the reciprocal of Q , see [2]. That is, if E is the projection on the range of Q , then

$$Q^{-1} Q = QQ^{-1} = E.$$

Here we choose the center of quadric curvatures at a point of (3.2) so that, it is at the shortest distance from γ .

III. When the rank of Q is k and the quadric does not have centers, then we say that f does not have a center of quadric curvature.

4. DIRECTION OF QUADRIC CURVATURE

In part I and II of section 3 we respectively call the vectors ξ and ξ_k the directions of quadric curvature of f at (c_1, \dots, c_n) . In III of section 3, we define the direction of quadric curvature to be a vector δ which satisfies

$$\delta = \delta E = - \left(\frac{\partial f}{\partial \xi} \right) EQ^{-1},$$

where E is the projection described in section 3.

5. VERTEX POINTS

Let at the point $\gamma = (c_1, \dots, c_n)$ of f the direction of quadric curvature be the same as the normal to $f = 0$. Then γ is called a vertex point of the function f .

Theorem: A necessary and sufficient condition for a point to be a vertex point of the function f is that at that point

$$PQ = QP,$$

where P and Q are the matrices described in section 3.

Proof: At a vertex point the projection of the direction of quadric curvature on the tangent plane is zero. Thus

$$-\left(\frac{\partial f}{\partial \xi}\right) Q^{-1} (I - P) = 0.$$

This implies that

$$Q^{-1} P Q = P.$$

In all cases this implies

$$P Q = Q P.$$

A vertex point in particular may become a spherical point, i.e. a point where

$$\frac{\partial^2 f}{\partial x_i \partial \bar{x}_j} = \lambda \delta_{ij}, \lambda$$

is a constant.

A vertex point will be called a cylindrical point when

$$\frac{\partial^2 f}{\partial x_i \partial \bar{x}_j} = \lambda \delta_{ij}, i, j \leq k,$$

$$\frac{\partial^2 f}{\partial x_i \partial \bar{x}_j} = 0, i, j > k.$$

6. FUNCTIONS OF FIXED CENTER

An interesting fact about these functions is that they are not necessarily quadrics.

The equation.

$$\xi Q = -\left(\frac{\partial f}{\partial \xi}\right) \tag{6.1}$$

where $\xi = (c_1 - x_1, \dots, c_n - x_n)$, and (c_1, \dots, c_n) is the fixed center gives f . To produce a counter example we let the origin be the center and the dimension of the space be two. Then in the real case (6.1) becomes

$$\left. \begin{aligned} x \frac{\partial^2 f}{\partial x^2} + y \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial f}{\partial x} \\ x \frac{\partial^2 f}{\partial x \partial y} + y \frac{\partial^2 f}{\partial y^2} &= \frac{\partial f}{\partial y} \end{aligned} \right\} \quad (6.2)$$

We can easily find a solution of (6.2) which is not a quadric.
For example

$$f = \frac{x^2}{2} \log \left(\frac{\sqrt{x^2 + y^2} + y}{x} \right) + \frac{y}{2} \sqrt{x^2 + y^2}.$$

REFERENCES

- [1] A. R. AMIR-MOÉZ, Quadrics in a Unitary space, *L'Enseignement Mathématique*, tome VII (1961), pp. 252-257.
- [2] PENROSE, A generalized inverse of matrices, *Proc. Cambridge Philos. Soc.*, vol. 51 (1953), pp. 406-413.
- [3] Ali R. AMIR-MOÉZ, Les sommets d'une surface, *L'Enseignement Mathématique* 10 (1964) p. 255

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