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II. Let the rank of  $Q$  be  $k$ , and centers exist. Then these centers are solutions of

$$\xi_k = \xi E = - \left( \frac{\partial f}{\partial \xi} \right) EQ^{-1}, \quad (3.2)$$

where  $Q^{-1}$  is the reciprocal of  $Q$ , see [2]. That is, if  $E$  is the projection on the range of  $Q$ , then

$$Q^{-1} Q = QQ^{-1} = E.$$

Here we choose the center of quadric curvatures at a point of (3.2) so that, it is at the shortest distance from  $\gamma$ .

III. When the rank of  $Q$  is  $k$  and the quadric does not have centers, then we say that  $f$  does not have a center of quadric curvature.

#### 4. DIRECTION OF QUADRIC CURVATURE

In part I and II of section 3 we respectively call the vectors  $\xi$  and  $\xi_k$  the directions of quadric curvature of  $f$  at  $(c_1, \dots, c_n)$ . In III of section 3, we define the direction of quadric curvature to be a vector  $\delta$  which satisfies

$$\delta = \delta E = - \left( \frac{\partial f}{\partial \xi} \right) EQ^{-1},$$

where  $E$  is the projection described in section 3.

#### 5. VERTEX POINTS

Let at the point  $\gamma = (c_1, \dots, c_n)$  of  $f$  the direction of quadric curvature be the same as the normal to  $f = 0$ . Then  $\gamma$  is called a vertex point of the function  $f$ .

*Theorem:* A necessary and sufficient condition for a point to be a vertex point of the function  $f$  is that at that point

$$PQ = QP,$$

where  $P$  and  $Q$  are the matrices described in section 3.

Proof: At a vertex point the projection of the direction of quadric curvature on the tangent plane is zero. Thus

$$-\left(\frac{\partial f}{\partial \xi}\right) Q^{-1} (I - P) = 0.$$

This implies that

$$Q^{-1} P Q = P.$$

In all cases this implies

$$P Q = Q P.$$

A vertex point in particular may become a spherical point, i.e. a point where

$$\frac{\partial^2 f}{\partial x_i \partial \bar{x}_j} = \lambda \delta_{ij}, \lambda$$

is a constant.

A vertex point will be called a cylindrical point when

$$\frac{\partial^2 f}{\partial x_i \partial \bar{x}_j} = \lambda \delta_{ij}, i, j \leq k,$$

$$\frac{\partial^2 f}{\partial x_i \partial \bar{x}_j} = 0, i, j > k.$$

## 6. FUNCTIONS OF FIXED CENTER

An interesting fact about these functions is that they are not necessarily quadrics.

The equation.

$$\xi Q = -\left(\frac{\partial f}{\partial \xi}\right) \tag{6.1}$$

where  $\xi = (c_1 - x_1, \dots, c_n - x_n)$ , and  $(c_1, \dots, c_n)$  is the fixed center gives  $f$ . To produce a counter example we let the origin be the center and the dimension of the space be two. Then in the real case (6.1) becomes