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EXISTENCE AND APPROXIMATION THEOREMS FOR ORDINARY DIFFERENTIAL EQUATIONS AND THEIR SYSTEMS [1])

PART I

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1. Historical background and weaknesses of Peano's theorem. Euler in his work written in Latin entitled " Institutionum Calculi Integralis " published in 1768, described a method of approximate solution of ordinary differential equations of first order

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

subjected to the initial condition $x = x_0$, $y = y_0$.

In this method, through the use of the recurrence formula

$$y_{i+1} = y_i + (x_{i+1} - x_i) f(x_i, y_i) \tag{2}$$

starting with the known pair of numbers (x_0, y_0) or the point $P_0(x_0, y_0)$ and an increasing sequence $\{x_i\}$, the numbers $y_1, \ldots, y_n$ or the points $P_i(x_i, y_i)$, $(i = 1, \ldots, n)$ are determined.

Euler considered these numbers y_i as approximations to the exact values $Y(x_i)$, where $y = Y(x)$ is assumed to be a solution of (1) satisfying the initial condition.

The geometric figure obtained by joining consecutively the points $P_0, P_1, \ldots, P_n$ by a straight line segment is called an " Euler polygonal-line " and his numerical procedure for the approximate solution of differential equations the Euler's Method. It is also quite often referred to as Cauchy-Lipschitz

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Method because of the improvements and generalization introduced by these mathematicians.

Indeed, although Euler originated this method, he never treated rigorously the convergence of the sequences and series involved in his process*) nor did he show the existence of the solution $Y(x)$ which he supposedly was approximating.

Between the years of 1820 and 1830, Cauchy in his lectures at the " Ecole Polytechnique " in Paris, brought in the mathematical rigor that Euler's original numerical procedure lacked and converted it into an existence and uniqueness theorem for equations (1) and for systems of differential equations of the type

$$\frac{dy_i}{dx} = f_i(x, y_1, \ldots, y_n) \quad (i = 1, \ldots, n). \tag{3}$$

A summarized form of this theorem was first published in 1835. A better but still incomplete version of it was published in 1844 by one of Cauchy's pupils, father Moigno (see [1]).

In 1868, Lipschitz considerably simplified Cauchy's proof [2]. In particular he replaced the requirement of the existence and the continuity of the partial derivative of $f(x, y)$ with respect to y for equation (1) and the existence and the continuity of the partial derivatives of the functions $f_i(x, y_1, \ldots, y_n)$ with respect to dependent variables y_i for the systems (3) by a less restrictive requirement which is now well known as a Lipschitz condition.

In 1886, Peano, separating the problem of the existence of the integral curves of (1) from that of their uniqueness at a point (x_0, y_0), established an existence theorem [3] with no other requirement on $f(x, y)$ except its continuity, which theorem he later extended to systems (3) [4].

Peano proved his existence theorem with the use of a class of formulas satisfying a certain approximation condition. He considered Euler's formula as an example and showed that it was a member of his class of formulas. Unfortunately, he failed

*) See for instance, " Leonardi Euleri Opera Omnia; Series 1, Opera Mathematica, Volumen XI; Institutiones Calculi Integralis; 1913 ", pp. 427-429.

to show what the other formulas were or how one could construct them.

Thus although Peano did establish, theoretically, the existence of a major and a minor solution through a point, he did not provide the practical means of constructing or approximating them.

He illustrated his findings with the example, by now well known, that $y = \pm x^2$ were the two extremal solutions of the differential equation

$$\frac{dy}{dx} = \begin{cases} \dfrac{4x^3 y}{x^4 + y^2} & \text{when } x \neq 0 \quad \text{or} \quad y \neq 0 \\ 0 & \text{when } x = y = 0 \end{cases}$$

corresponding to the initial conditions $x = 0$, $y = 0$. But he did not indicate which were the members of his class of formulas that would reveal the existence of these specific extremal solutions or enable one to approximate them. Euler's polygonal lines defined through Euler's recurrence relation, as used by Peano, would show only the existence of the trivial solution $y = 0$.

It is worth mentioning at this stage, fleetingly, leaving a more thorough discussion of this subject for the second part of this work, that if the polygonal lines were defined, for instance, with the use of the formula

$$y_{i+1} = y_i + \frac{1}{2}(k_{i,0} + k_{i,1})$$

where

$$k_{i,0} = hf(x_i + \tfrac{1}{2}h^{\frac{1}{2}}, y_i + \tfrac{1}{2}h)$$

$$k_{i,1} = hf(x_i + h, y_i + k_{i,0})$$

then one would obtain the major solution to the right of the point (0, 0) and the minor one to the left.

A decade later Arzelà [5] considerably simplified the proof of Peano's theorem by using solely Euler's recurrence relation without recourse to Peano's class of formulas.

However, in doing so, Arzelà was reducing, perhaps unwittingly, the theoretical scope of Peano's theorem and transforming it into a constructive existence and pseudo-uniqueness theorem, since, as will be shown later on, the existence of only a single solution can be established and approximated through this method.

Nevertheless, Arzelà's method and its variations [6, 7, 8, 9] are today preferably used in place of, and even at times mistaken for, Peano's original, tedious proof.

However in spite of this wide and universal acceptance of Arzelà's method there seems ground to believe that the before mentioned subtle limitation imposed by this method remains either undetected or misunderstood.

More recently Sansone [10], through the adaptation of an elegant method given by Tonelli, relative to the existence of solutions of Volterra type functional equations [11], still further shortened and simplified the proof of Peano's theorem.

It must be noted once more, as in Arzelà's proof, that the resulting simplicity was detrimental to the theoretical scope of Peano's theorem.

The Sansone-Tonelli method also yields and approximates only one solution from an infinite number of solutions that may pass through a point. However in this case, this solution may or may not be the same as the one obtained through Arzelà's method. This, as will be seen, is due to the fact that Arzelà's and Sansone-Tonelli's methods are each biased in favor of one particular solution, not necessarily the same, from all the solutions of the differential equation(s) satisfying the given initial condition.

Furthermore, this " biased " solution may not be an interesting extremal solution. In this case these two methods do not provide means even for the approximate determination of this preferred (extremal) solution.

Although Arzelà's method is more convenient in practical use than Sansone-Tonelli's, both are inadequate for numerical evaluation. Besides, from the standpoint of approximate solution of differential equations, Arzelà's method [9b, 12a] is evidently no other than the numerical procedure given by Euler some 130 years earlier, which as is well known, is a " first order approximation " method, that is, a very poor one.

At times Peano's theorem is treated by the method of Weierstrass' polynomial approximation originated by Severini [13, 12b, 14]. But this method does not provide any practical basis for the approximate solution of the differential equations.

Finally during the period 1946-1947 Baiada [15, 16, 17] and Cafiero [18], simultaneously but independently of each other, succeeded in improving Tonelli-Sansone's and Arzela's versions of Peano's theorem, respectively. Their approach made possible the determination of every solution through the initial point P_0.

Baiada's method, like Tonelli-Sansone's requires an integration at each step. This makes the process impractical except in some special cases, such as, when $f(x, y)$ is a polynomial.

In Cafiero's method the rectangular region $V_1 V_2 V_3 V_4$ (fig. 1) is partitioned by a net of horizontal and vertical lines into a set of rectangles which are called " partial rectangles ". A point P belonging to the region $V_1 V_2 V_3 V_4$ will fall in $R^{(h)}(P)$, a certain partial rectangle corresponding to some partitioning $D^{(h)}$ (where " h " designates the norm of partitioning of the interval $[x_0, b_1]$ of figure 1).

A number $\mu^{(h)}(P)$ satisfying the relation

$$m^{(h)}(P) \leqq \mu^{(h)}(P) \leqq M^{(h)}(P)$$

is associated with the point P, $m^{(h)}(P)$ and $M^{(h)}(P)$ being the extremal values of $f(x, y)$ on $R^{(h)}(P)$.

A line with slope $\mu^{(h)}(P_0)$ is drawn from P_0 to $P_{1,h}$, its intersection point with the right vertical side of $R^{(h)}(P_0)$. The process is repeated again and again until a polygonal line is obtained extending over the entire interval $[x_0, b_1]$. Thereafter the polygonal lines corresponding to all modes of partitioning $D^{(h)}$ are considered and Ascòli's theorem is used.

These are some of the main features of the proof of Cafiero's version of Peano's theorem. They have been indicated here for two reasons: this approach is little known and it is entirely inadequate for concrete practical applications.

It is now our purpose to establish a strong constructive theorem.

In this theorem also the continuity of the function $f(x, y)$ [19, 20] will be the sole requirement, however the polygonal lines will be defined by a class of parametric recurrence formulas instead of the single and rigid formula (2). In this way the various weaknesses or shortcomings encountered in the Peano's theorem will be avoided. However the advantages of this new existence theorem will not be made evident until the second part of this work, where it will also be shown that only a change of the recurrence relation through which the (convergent) sequence of polygonal lines $\{\Gamma_i\}$ is defined may affect the integral curve Γ, the limit of $\{\Gamma_i\}$.

2. Symbols and Notations. For the sake of convenience and unless otherwise stated the symbols P_i, $\tilde{P}$, $\bar{P}$, P' and P'' will represent the points $P_i\,(_i, y_i)$ $\tilde{P}\,(\tilde{x}, \tilde{y})$, $\bar{P}\,(\bar{x}, \bar{y})$, $P'\,(x', y')$ and $P''\,(x'', y'')$ respectively.

The function $f(x, y)$ and its value at P_i will be often denoted by f and f_i, respectively.

The first order derivative of a function $\varphi(x)$ will be denoted by $\frac{d\varphi}{dx}$ and whenever there is no confusion, merely by φ'. Its higher order derivatives will be denoted by $D_x^n\,\varphi(x)$, $n \geqq 2$, and whenever there is no ambiguity, merely by $D^n\,\varphi(x)$ or $D^n\,\varphi$.

As usual, $\varphi(x) \in C^n$ will indicate that $\varphi(x)$ belongs to the class C^n, that is, φ and its derivatives up to the order n (inclusive), are continuous.

Finally the closure of the open set X will be designated by $\bar{X}$.

3. We shall first be concerned with the derivation of four lemmas. Consider the differential equation (1) where $f(x, y)$ is a single but real valued function of the two real variables x, y, defined and continuous on some open and simply connected region R of the x, y plane.

Let $P_0\,(x_0, y_0)$ be a point of R. There exists a circular neighborhood Ω_1 of P_0 such that $\bar{\Omega}_1$ is interior to R. Disregarding the case $f(x, y) \equiv 0$ as trivial, assume m represents the norm of $f(x, y)$ over the set $\bar{\Omega}_1$, that is

$$\|f(x, y)\|_{\bar{\Omega}_1} = m > 0 .$$

Thus for all points of $\bar{\Omega}_1$ we have $|f(x, y)| \leqq m$.

The so-called butterfly region shall now be constructed in the usual way as follows:

Draw through P_0 the lines L_1 and L_2 with respective slopes $-m$ and $+m$. These lines intersect the circumference constituting the boundary of the circle Ω_1 in four points, two to the left of P_0 and two to the right.

For the sake of simplicity, only the two points to the right will be considered, henceforth designated by V_1 and V_2 as shown in Figure 1.

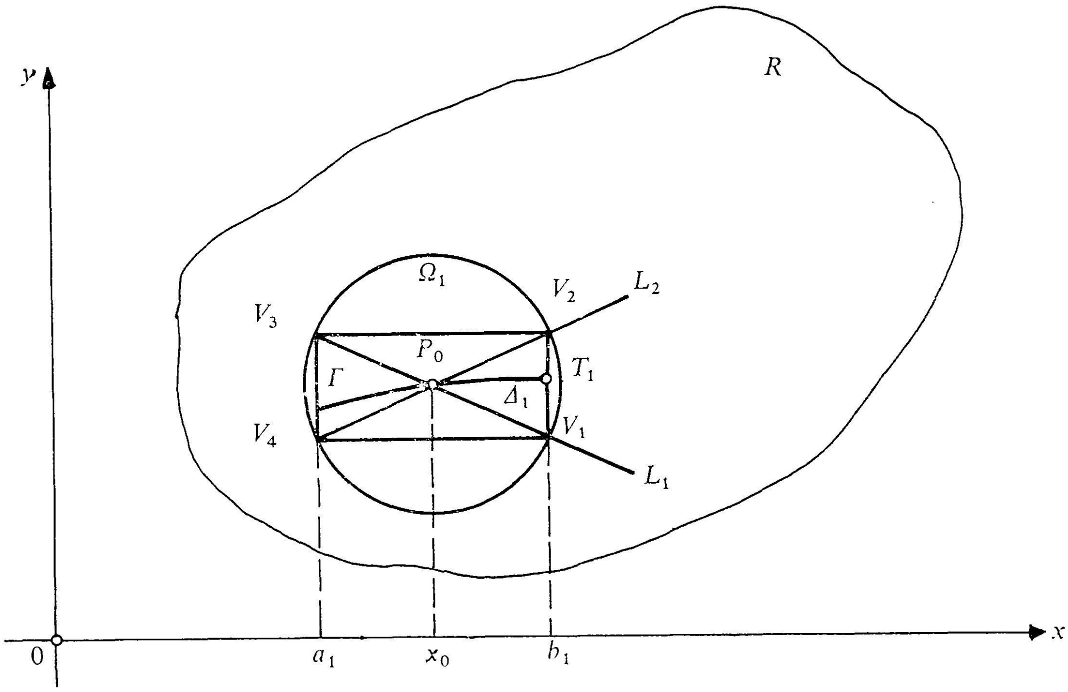

Fig. 1

The line through V_1 and V_2, which intersects perpendicularly the x-axis at $(b_1, 0)$, together with the lines L_1 and L_2 bound $\varDelta_1$, the triangular region $P_0\, V_1\, V_2$ which with its boundary lie entirely in the interior of R.

What follows will revolve mainly around the closed interval $I_1 = [x_0, b_1]$ and the closure $\bar{\varDelta}_1$ (the right half of the butterfly region).

We shall refer to I_1 as the " first right partial interval of convergence " or merely as an " interval of convergence " while $\bar{\varDelta}_1$ will be referred to as the " first right partial region of convergence " or simply a " region of convergence ".

Consider an arbitrary partition of I_1 by dividing this interval in any manner into n parts. Let the points of division be

$$x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b_1$$

We shall now describe a device which to every division point of I_1 will make correspond a unique but special point in $\overline{\Delta}_1$.

Let x_j and x_{j+1} be any two consecutive points of the partition and define $x_{j+1} - x_j = h_j \leqq H(n)$, the latter being the norm of the partition.

Assume the point $P_j(x_j, y_j) \in \overline{\Delta}_1$ corresponding to x_j to be known. Relative to this point P_j consider the coefficients

$$\begin{cases} k_{j,0} = h_j f(x_j + \alpha h_j^q, y_j + \beta h_j^{q'}) \\ k_{j,i} = h_j f(x_j + \mu_{j,i-1} h_j, y_j + \eta_{j,i-1} k_{j,i-1}) \quad i = 1, 2, \dots, p \end{cases} \tag{4}$$

where p is an arbitrarily selected natural number, "q"s are positive rationals, and the Greek letters α, β, μ and η designate parameters or arbitrary constants such that

$$\begin{cases} \alpha^2 (b_1 - x_0)^{2q-2} + \beta^2 (b_1 - x_0)^{2q'-2} \leqq 1, \; q > 0, \; q' > 0 \\ 0 \leqq \mu_{j,i-1} \leqq 1 \qquad i = 1, 2, \dots, p. \\ |\eta_{j,i-1}| \leqq 1 \end{cases} \tag{5}$$

Define

$$y_{j+1} = y_j + \sum_{i=0}^{p} \lambda_{j,i} k_{j,i} \tag{6}$$

where the "k"s are known quantities obtained as described above and the "λ"s designate again, just as the other Greek letters, parameters or arbitrary real constants but now such that

$$\sum_{i=0}^{p} \lambda_{j,i} = 1, \; \lambda_{j,i} \geqq 0. \tag{7}$$

Let the point $P_{j+1}(x_{j+1}, y_{j+1})$, where y_{j+1} is obtained by the use of any particular recurrence formula (6), correspond to the partition point x_{j+1}. We shall show that $P_{j+1} \in \overline{\Delta}_1$.

Let us recall that $|f(x, y)| \leqq m$ for all points of $\overline{\Omega}_1$. But because of the first relation in (5), $(x_j+\alpha h_j^q,\ y_j+\beta h_j^{q'}) \in \overline{\Omega}_1$, thus

$$|f(x_j+\alpha h_j^q\, y_j+\beta h_j^{q'})| \leqq m .$$

Multiplying both members of this relation by $h_j > 0$ we obtain

$$|k_{j,\,0}| \leqq mh_j . \tag{8}$$

On the other hand with $i = 1$ we have from (4)

$$k_{j,\,1} = h_j f(x_j+\mu_{j,\,0}\, h_j, y_j+\eta_{j,\,0}\, k_{j,\,0}) ,$$

or also

$$|k_{j,\,1}| = h_j\ |f(x_j+\mu_{j,\,0}\, h_j, y_j+\eta_{j,\,0}\, k_{j,\,0})| . \tag{9}$$

Furthermore, since $|\eta_{j,\,0}| \leqq 1$, one derives from the relation (8)

$$-mh_j \leqq \eta_{j,\,0}\, k_{j,\,0} \leqq mh_j$$

or

$$y_j - mh_j \leqq y_j + \eta_{j,\,0}\, k_{j,\,0} \leqq y_j + mh_j . \tag{10}$$

The triangular region Δ_1 is bounded by the lines $x = b_1$ and $y = y_0 \pm m\,(x-x_0)$. Thus if a point $P(x, y)$ belongs to $\overline{\Delta}_1$ then its coordinates must satisfy

$$x_0 \leqq x \leqq b_1$$

$$y_0 - m\,(x-x_0) \leqq y \leqq y_0 + m\,(x-x_0) . \tag{11}$$

Since $P_j \in \overline{\Delta}_1$, $x_0 \leqq x_j \leqq b_1$, then (11) becomes

$$y_0 - m\,(x_j-x_0) \leqq y_j \leqq y_0 + m\,(x_j-x_0) .$$

From the latter relation we derive

$$y_0 - m\,(x_{j+1}-x_0) \leqq y_j - mh_j$$

$$y_j + mh_j \leqq y_0 + m\,(x_{j+1}-x_0) .$$

In view of the fact that $x_{j+1} \leqq b_1$ these relations in turn can be replaced by

$$y_0 - m\,(b_1-x_0) \leqq y_j - mh_j \tag{12a}$$

$$y_j + mh_j \leqq y_0 + m\,(b_1-x_0) . \tag{12b}$$

The combination of the relations (10), (12a) and (12*b*) gives

$$y_0 - m(b_1 - x_0) \leqq y_j + \eta_{j,0} k_{j,0} \leqq y_0 + m(b_1 - x_0). \qquad (13)$$

The equations of the two parallel lines to the x-axis through the points V_1, $V_2 \in \bar{\Omega}_1$ are

$$y = y_0 \pm m(b_1 - x_0). \qquad (14)$$

A comparison between (13) and (14) indicates that the point $(x_j + \mu_{j,0} h_j, y_j + \eta_{j,0} k_{j,0})$ falls in $\bar{\Omega}_1$ between these parallel lines through V_1 and V_2. It follows that

$$|f(x_j + \mu_{j,0} h_j, y_j + \eta_{j,0} k_{j,0})| \leqq m.$$

In view of the above relation, (9) can be replaced by

$$|k_{j,1}| \leqq h_j m. \qquad (15)$$

Following the same procedure that enabled us to obtain the relation $|k_{j,1}| \leqq h_j m$ from $|k_{j,0}| \leqq h_j m$ we find consecutively

$$|k_{j,2}| \leqq h_j m$$

$$\cdots \qquad \cdots$$

$$|k_{j,p}| \leqq h_j m.$$

In other words we have

$$|k_{j,i}| \leqq h_j m \qquad i = 0, 1, \ldots, p \qquad (16)$$

From (4) we derive

$$|y_{j+1} - y_j| = |\sum_{i=0}^{p} \lambda_{j,i} k_{j,i}| \leqq \sum_{i=0}^{p} |\lambda_{j,i} k_{j,i}|.$$

Then on account of (7) and (16) we have $|y_{j+1} - y_j| \leqq h_j m$ and consequently also

$$y_j - h_j m \leqq y_{j+1} \leqq y_j + h_j m \qquad (17)$$

which indicates that the point $P_{j+1} \in \bar{\Delta}_1$ as was to be proved.

Thus any recurrence formula (6) enables us to depart from a known point $P_j \in \overline{\Delta}_1$ corresponding to x_j and to determine another point $P_{j+1} \in \overline{\Delta}_1$ corresponding to x_{j+1}.

Since P_0 is a known point we shall depart from it by taking $j = 0$ and determine through the repeated application of a particular formula (6) the sequence of points $P_0, P_1, \ldots, P_{n-1}, P_n$ all belonging to $\overline{\Delta}_1$ and corresponding to respective division points of the considered partition of I_1.

Joining these points consequtively with a line, a polygonal line $P_0 P_1 \ldots P_n$ is obtained having these same points as vertices and lying entirely in $\overline{\Delta}_1$ and therefore being bounded.

Let γ_n designate the polygonal line obtained through the partitioning of the interval I_1 arbitrarily, into n parts.

Carry out this process for $n = 1, 2, \ldots$ starting with the entire interval I_1 and by adding more and more partitioning points to the ones already existing, in such a way, that the norm $H(n)$ decreases monotonically to zero with increasing n.

Thus a sequence of polygonal lines $\{\gamma_n\}$ constituting a family $\mathcal{F}$ is obtained.

Now we are ready to establish four lemmas which exhibit some of the properties of these polygonal lines and are essential for later use.

For this purpose let $\gamma_n \in \mathcal{F}$ be an arbitrarily selected polygonal line with n sides. Thus any property possessed by γ_n can be attributed to all the polygonal lines constituting the family $\mathcal{F}$.

Lemma I: — The polygonal lines of $\mathcal{F}$ represent uniformly bounded functions of $x \in I_1$.

Proof: Since all the polygonal lines $\gamma_n \in \mathcal{F}$ lie in $\overline{\Delta}_1$ they are uniformly bounded and consequently the functions $\gamma_n(x)$ representing them are also uniformly bounded for all $x \in I_1$.

Lemma II: — For any two distinct points $P'(x', y')$ and $P''(x'', y'')$ of a polygonal line the relation

$$|y'' - y'| \leqq m\,|x'' - x'| \tag{18}$$

holds.

Proof: The coordinates of any two consecutive vertices $P_j\,(x_j, y_j)$, $P_{j+1}\,(x_{j+1}, y_{j+1})$ of γ_n satisfy (17), which relation can also be written:

$$mx_j - mx_{j+1} \leqq y_{j+1} - y_j \leqq mx_{j+1} - mx_j\,. \qquad (19)$$

For any two distinct points P', P'' of the segment or side $\overline{P_j P_{j+1}}$ we have

$$\frac{y'' - y'}{x'' - x'} = \frac{y_{j+1} - y_j}{x_{j+1} - x_j}\,. \qquad (20)$$

The combination of (19) and (20) gives us

$$mx' - mx'' \leqq y'' - y' \leqq mx'' - mx' \qquad (21)$$

which is essentially the same as the relation (19) except for the fact that the numbers (x_j, y_j) and (x_{j+1}, y_{j+1}) are replaced by (x', y') and (x'', y'').

We thus conclude that the relation (19) holds not only for the coordinates of any two consecutives vertices P_j, P_{j+1} of γ_n but also for the coordinates of any two distinct points of any one side of γ_n.

Furthermore (21) is equivalent to (18). Therefore the validity of the relation (18) for the coordinates of any two distinct points of a side of γ_n becomes established.

We shall now consider the case where the points P' and P'' belong to two different sides, for instance $\overline{P_j P_{j+1}}$ and $\overline{P_q P_{q+1}}$ of γ_n, evidently with integers $j \geqq 0$ and $j + 1 \leqq q \leqq n - 1$, and also for the sake of convenience, with the assumption that $x_j \leqq x' < x_{j+1}$ and $x_q < x'' \leqq x_{q+1}$.

The use of the relation (19) with the coordinates of the points P' and P_{j+1}, P_{j+1} and P_{j+2}, ..., P_q and P'' will give consecutively

$$mx' - mx_{j+1} \leqq y_{j+1} - y' \leqq mx_{j+1} - mx'$$

$$mx_{j+1} - mx_{j+2} \leqq y_{j+2} - y_{j+1} \leqq mx_{j+2} - mx_{j+1}$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$mx_q - mx'' \leqq y'' - y_q \leqq mx'' - mx_q\,.$$

The addition of member to member, respectively, of all these relations, yields

$$mx' - mx'' \leqq y'' - y' \leqq m'' - mx'$$

that is

$$| y'' - y' | \leqq m | x'' - x' |$$

which proves the lemma.

However it must be observed that instead of m, if we use m_1 and m_2, the actual minimum and maximum values respectively of $f(x, y)$ on $\bar{\Omega}_1$, then the above process would yield

$$m_1 (x'' - x') \leqq y'' - y' \leqq m_2 (x'' - x') \qquad (22)$$

Lemma III: — The family $\mathscr{F}$ is composed of equicontinuous functions of x on the closed interval I_1.

Proof: Let $y = \gamma_n(x)$ be the equation of $\gamma_n \in \mathscr{F}$ and $\in$ be a given positive number. Take $\delta = \frac{\in}{m}$ ·

Assume P' and P'' be any two distinct points on γ_n but such that $| x'' - x' | < \delta$.

Then one has

$$m | x'' - x' | < \in .$$

Furthermore from the consideration of the Lemma II, it follows that

$$| y'' - y' | < \in ,$$

or

$$| \gamma_n(x'') - \gamma_n(x') | < \in$$

where $x', x'' \in I_1$.

Thus since for every positive number $\in$ there is a positive number $\delta = \in / m$ such that whenever $| x'' - x' | < \delta, x', x'' \in I_1$ the inequality

$$| \gamma_n(x'') - \gamma_n(x') | < \delta \qquad n = 1, 2, \ldots$$

holds, it results that the polygonal line functions of $\mathscr{F}$, that is $\gamma_n(x)$, $(n = 1, 2, \ldots)$, are equicontinuous functions on I_1.

Lemma IV: — From the sequence of polygonal line functions $\{\gamma_n(x)\}$, $x \in I_1$ and $n = 1, 2, \ldots$, it is possible to select a uniformly convergent subsequence.

Proof: From lemmas I and III it is known that the polygonal line functions $\gamma_n(x) \in \mathcal{F}$ are uniformly bounded and are equicontinuous on the closed interval $I_1 = [x_0, b_1]$. Then in accordance with Ascoli's Theorem it is possible to extract from $\{\gamma_n(x)\}$ a subsequence $\{\gamma_{n_i}(x)\}$ which will converge uniformly to a function $\Gamma(x)$. And since $\{\gamma_n(x)\}$ are continuous on the interval I_1, the same is true also for this limit function $\Gamma(x)$.

For the sake of simplicity the convergent subsequence $\{\gamma_{n_i}(x)\}$ will be henceforth designated by $\{\Gamma_i(x)\}$ with $i = 1, 2, \ldots$; but the subscript i does not necessarily imply, anymore, that the interval $I_1 = [x_0, b_1]$ has been divided into i parts since now $i \leqq n$.

Furthermore, the norm of the partitioning of the interval I_1 relative to a polygonal line Γ_i, shall be designated by H_i.

4. We are now in the position to prove that the limit function is differentiable and also satisfies the differential equation (1); that is, Γ is an integral curve.

Let x' be an arbitrarily chosen fixed point from the closed interval $I_1 = [x_0, b_1]$ and let x'' be a neighboring point. It must be shown that

$$\lim_{x'' \to x'} \frac{\Gamma(x'') - \Gamma(x')}{x'' - x'} = f(x', \Gamma(x')).$$

In other words, we must show that for each $\epsilon > 0$, a natural number N and a real number $\delta > 0$ exist, such that

$$\frac{\Gamma_\nu(x'') - \Gamma_\nu(x')}{x'' - x'} - f(x', \Gamma(x'))| < \epsilon$$

for all $\nu > N$ and whenever $0 < |x'' - x'| < \delta$.

Let us observe that on account of the continuity of the function $f(x, y)$ on the closed circular region $\bar{\Omega}_1$ and particularly at the point $P'(x', y') \in \bar{\Delta}_1$ where $y' = \Gamma(x')$, to an arbitrarily selected $\epsilon > 0$, there corresponds a circular neighborhood $\mathcal{C}$ of P', $\mathcal{C} \subset \Delta_1$ such that for all points $P(x, y) \in \mathcal{C}$ one has

$$|f(x, y) - f(x', y')| < \epsilon.$$

Consider through P' two lines with slope $\pm \frac{5}{2} m$. These lines intersect the circle bounding $\mathcal{C}$ in four points. Joining these points consecutively with a straight line a rectangular open region $\mathscr{R} \subset \mathscr{C} \subset \Delta_1$ is obtained.

If 4δ designates the length of a horizontal side of $\mathscr{R}$ then $10\delta m$ will be the length of a vertical side (see figure 2).

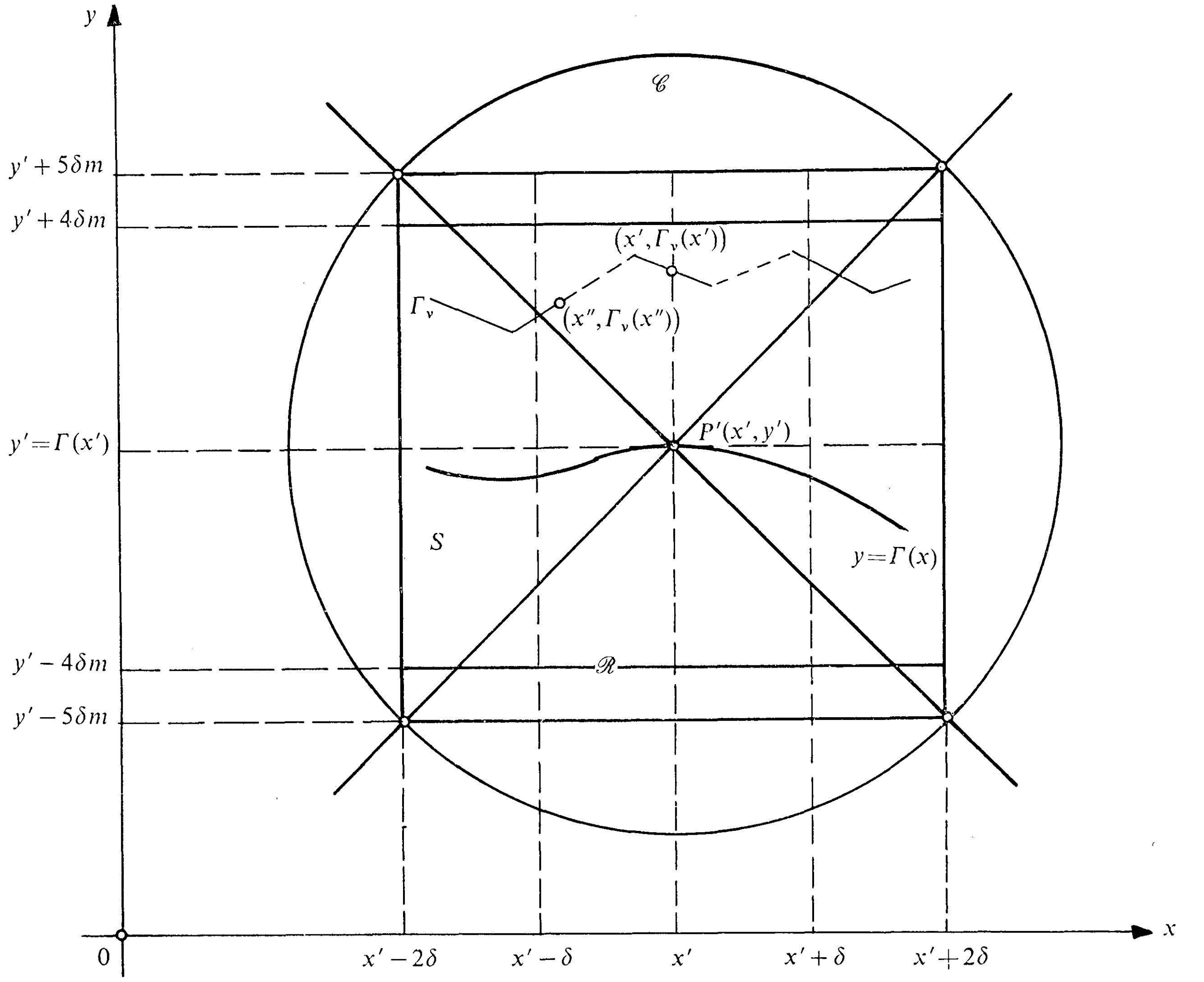

Fig. 2

Let S represent that subset of $\mathscr{R}$ which is the rectangular region bounded by the lines $y = y' \pm 4\delta m$.

Evidently for all points $P\,(x, y) \in \bar{S}$ the preceding inequality holds true. Thus we can say that given $\in > 0$ there exists a $\delta > 0$ such that

$$|f(x, y) - f(x', y')| < \in \qquad (23a)$$

whenever

$$|x - x'| \leqq 2\delta \text{ and } |y - y'| \leqq 4\delta m . \qquad (23b)$$

Now since the sequence of functions $\{\Gamma_i(x)\}$ converges uniformly to $\Gamma(x)$ on I_1, if $\epsilon' = \min(\epsilon, 2\delta m)$ then we know that there exists a natural number N' such that for all $\nu > N'$

$$|\Gamma(x) - \Gamma_\nu(x)| < \epsilon'$$

on the interval I_1.

Evidently this inequality holds also for $x' \in I_1$, thus

$$|\Gamma(x') - \Gamma_\nu(x')| < 2\delta m . \tag{24}$$

On a polygonal line Γ_ν $(\nu > N')$ consider the points $(x', \Gamma_\nu(x'))$ and $(x, \Gamma_\nu(x))$ with $|x - x'| < 2\delta$. Then because of relation (18) of Lemma II one has

$$|\Gamma_\nu(x) - \Gamma_\nu(x')| < 2\delta m . \tag{25}$$

We can write

$$|\Gamma_\nu(x) - \Gamma(x')| \leqq |\Gamma_\nu(x) - \Gamma_\nu(x')| + |\Gamma_\nu(x') - \Gamma(x')| .$$

The consideration of (24) and (25) gives then

$$|\Gamma_\nu(x) - \Gamma(x')| < 4\delta m$$

provided that $\nu > N'$ and $|x - x'| < 2\delta$.

These inequalities imply in their turn that the point $(x, \Gamma(x))$ belongs to the region S. In other words, those portions of the polygonal lines Γ_ν, $\nu > N'$, that fall between the two lines $x = x' \pm 2\delta$ lie entirely in the interior of the rectangular region S.

Now it must be observed that H_n, the norm of the partition of the interval I_1 decreases monotonically as the number of division or partition points increases. It follows that there exists a natural number $N \geqq N'$ such that for all the polygonal lines Γ_ν, $\nu > N$, $H_\nu > \delta$.

This requirement implies that any polygonal line Γ_ν must have in S at least one vertex point lying to the left of the line $x = x' - \delta$, one to the right of $x = x' + \delta$, and at least two between these lines.

Consider that part of the polygonal line Γ_ν, $\nu > N$, which extends, from the first vertex located to the left of the line

$x = x' - \delta$ to the other first vertex located to the right of $x = x' + \delta$.

Since now $h_j \leqq H_\nu < \delta$, the relations (14) give

$$- \delta m < k_{j,i} < \delta m \qquad i = 0, 1, \ldots, p .$$

Then relative to the points $(x_j + \mu_{j,\,i-1}\, h_j, y_j + \eta_{j,\,i-1}\, k_{j,\,i-1})$ associated with a vertex $(x_j, \Gamma_\nu(x_j))$, of the above described part of any one of these polygonal lines one has

$$y_j - \delta m \leqq y_j + \eta_{j,\,i-1}\, k_{j,\,i-1} \leqq y_j + \delta m .$$

As a consequence of these relations it is seen that the considered points $(x_j + \mu_{j,\,i-1}\, h_j, y_j + \eta_{j,\,i-1}\, k_{j,\,i-1})$ may lie not only in S but fall also in its complement in $\mathscr{R}$ which consists of two rectangular strips of height δm on both sides of S, below and above it.

Then if $\overline{m}_1 \geqq m_1$ and $\overline{m}_2 \leqq m_2$ are the two extreme values of the continuous function $f(x, y)$ on $\overline{\mathscr{R}}$, we must have

$$\overline{m}_1 \leqq f(x_j + \mu_{j,\,i-1}\, h_j, y_j + \eta_{j,\,i-l}\, k_{j,\,i-l}) \leqq \overline{m}_2 .$$

It follows that relative to two consecutive vert ces $(x_j, \Gamma_\nu(x_j))$, $(x_{j+1}, \Gamma_\nu(x_{j+1}))$ the formula (6) yields

$$\overline{m}_1 \leqq \frac{\Gamma_\nu(x_{j+1}) - \Gamma_\nu(x_j)}{x_{j+1} - x_j} \leqq \overline{m}_2 .$$

In other words the slope of each side of the considered part of the polygonal lines Γ_υ is bounded by the numbers $\overline{m}_1$ and $\overline{m}_2$.

Therefore, an almost verbatim repetition of the proof of Lemma II applied to the point $(x', \Gamma_\nu(x'))$ and $(x'', \Gamma_\nu(x''))$ on the same polygonal line such that $0 < |x'' - x'| < \delta$, yields

$$\overline{m}_1 (x'' - x') \leqq \Gamma_\nu(x'') - \Gamma_\nu(x') \leqq \overline{m}_2 (x'' - x')$$

or

$$\overline{m}_1 \leqq \frac{\Gamma_\nu(x'') - \Gamma_\nu(x')}{x'' - x'} \leqq \overline{m}_2 \qquad (26)$$

Let $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$ be the points of $\bar{\mathcal{R}} \subset \bar{\mathcal{C}}$ where $f(x, y)$ takes its extreme values $\bar{m}_1$ and $\bar{m}_2$, respectively. Then on account of the relations (23a) and (23b) one has:

$$|f(\bar{x}_1, \bar{y}_1) - f(x', y')| < \epsilon$$

$$|f(\bar{x}_2, \bar{y}_2) - f(x', y')| < \epsilon ;$$

that is

$$f(x', y') - \epsilon < \bar{m}_1 < f(x', y') + \epsilon \tag{27}$$

$$f(x', y') - \epsilon < \bar{m}_2 < f(x', y') + \epsilon . \tag{28}$$

The combination of (27) and (28) with (26) yields

$$f(x', y') - \epsilon < \frac{\Gamma_\nu(x'') - \Gamma_\nu(x')}{x'' - x'} < f(x', y') + \epsilon .$$

Noting that $y' = \Gamma(x')$ we can also write

$$\left| \frac{\Gamma_\nu(x'') - \Gamma_\nu(x')}{x'' - x'} - f(x', \Gamma(x')) \right| < \epsilon$$

with $\nu > N$ and $0 < |x'' - x'| < \delta$.

On observing that whatever has been established to the right of P_0 can in similar manner be established to the left of P_0 (taking $h_j < 0$), we can announce:

General Existence Theorem: Consider the ordinary differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

where $f(x, y)$ is a single but real valued function of the two real variables x, y, defined and continuous on some open and simply connected region R of the x, y-plane. Then for each point $P_0(x_0, y_0) \in R$ there exist a closed interval $I_1 = [a_1, b_1]$, $a_1 < x_0 < b_1$ and a function $\Gamma(x) \in C^1$ on I_1 such that $\Gamma(x)$ is a solution of (1) on I_1 and $y_0 = \Gamma(x_0)$.

5. In order to consider the case where P_0 is a boundary point of R we assume now that $f(x, y)$ is defined and continuous on the union $\tilde{R}$ of the open region R and some set of its boundary points.

Let ω, with radius ρ and center at P_0, be a circular arc extending from one boundary point of R to another and lying entirely in R.

This arc divides $\tilde{R}$ into two sets, one of which contains P_0. Let Ω designate the union of this set and ω.

Suppose that the following conditions are realized (if necessary by decreasing the radius ρ):

a) The numbers $\tilde{m}_1 = \inf \{f(x, y)\}$ and $\tilde{m}_2 = \sup \{f(x, y)\}$, $(x, y) \in \Omega$, are finite;

b) Two points $\tilde{P}_1$ and $\tilde{P}_2$ with common abscissa $\tilde{x}$ can be found on the lines $\tilde{L}_1$ and $\tilde{L}_2$, $y = y_0 + (x - x_0)\, \tilde{m}_1$ and $y = y_0 + (x - x_0)\, \tilde{m}_2$, respectively, such that the segments $P_0 \tilde{P}_1$ and $P_0 \tilde{P}_2$ lie in Ω.

Let $\tilde{\Delta}$ represent the closure of the triangular region $P_0\, \tilde{P}_1\, \tilde{P}_2$ bounded by the lines $\tilde{L}_1$, $\tilde{L}_2$ and $x = \tilde{x}$.

Clearly in $\tilde{\Delta} \subset \Omega$, we can proceed just as before in $\tilde{\Delta}_1$, and show the existence of an integral curve Γ starting at P_0 and lying entirely in $\tilde{\Delta}$.

It must be mentioned that there are other methods of construction of such triangular regions $\tilde{\Delta}$ besides the above indicated procedure.

It is left to the interested reader to show that regions of convergence $\tilde{\Delta}$ can easily be found for the differential equation

$$\frac{dy}{dx} = \frac{x - \sqrt{2x^2 - 2y^2}}{y}$$

with P_0 (1,1).

6. Substitutes for the Set of Incremental Coefficients k: Without any change in the method of proof of this general existence theorem, the set of coefficients (4) can be replaced by the following larger set of coefficients

$$\begin{aligned} k_{j,0} &= h_j f(x_j + \alpha h_j^q, y_j + \beta h_j^{q'}) \\ k_{j,i} &= h_j f(x_j + \mu_{j,i-1} h_j, y_j + \sum_{i'=0}^{i-1} \eta_{j,i-1,i'} k_{j,i'}), \qquad (29) \\ i &= 1, 2, \ldots, p \end{aligned}$$

with the parameters α, β, μ and η satisfying the relations

$$\alpha^2 (b_i - x_0)^{2q-2} + \beta^2 (b_1 - x_0)^{2q'-2} \leqq 1 , \qquad q > 0, q' > 0$$

$$0 \leqq \mu_{j, i-1} \leqq 1 \qquad i = 1, 2, \ldots, p$$

$$\sum_{i'=0}^{i-1} | \eta_{j, i-1, i'} | \leqq 1 .$$

In fact, proceeding as before it will be readily found that the relations (16), that is,

$$| k_{j, i} | \leqq h_j m \qquad i = 0, 1, \ldots, p$$

hold, the four lemmas are valid, as are other relations and argumentations given in the course of the proof of the theorem.

It is seen that set (4) is a subset of (29).

We readily recognize that not only Euler's relation but also other wellknown recurrence formulas, like those used in Modified Euler, Runge-Kutta and Nyström methods, are members of this consistent set of incremental coefficients [21].

Finally it is worthwhile observing that the " k "s which appear in (4) and (29) are in linear combinations. However with some slight and obvious changes we may take them in non-linear combinations. One such set of non-linear type is:

$$k_{j, 0} = h_j f(x_j + \alpha h_j^q, y_j + \beta h_j^{q'})$$

$$k_{j, i} = h_j f\left(x_j + \mu_{j, i-1} h_j, y_j + \left(\sum_{i'=0}^{i-1} \eta_{j, i-1, i'} k_{j, i'}^r \right)^{\frac{1}{r}} \right), \quad (30)$$

$$i = 1, 2, \ldots, p$$

where r is a natural number and the parameters α, β, μ and η satisfy the relations

$$\alpha^2 (b_1 - x_0)^{2q-2} + \beta^2 (b_1 - x_0)^{2q'-2} \leqq 1 \qquad q > 0, q' > 0$$

$$0 \leqq \mu_{j, i-1} \leqq 1$$

$$\sum_{i'=0}^{i-1} \eta_{j, i-1, i'} \leqq 1 , \qquad \eta_{j, i-1, i'} \geqq 0 .$$

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