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for some coherent sheaf $\mathscr{I}$ of ideals of $\mathcal{O}_X$. An *open analytic subspace* of $(X, \mathcal{O}_X)$ is just a restriction $(U, \mathcal{O}_X \mid U)$, $U$ open in $X$. An *analytic subspace* of an analytic space $(X, \mathcal{O}_X)$ is a closed analytic subspace $(Y, \mathcal{O}_Y)$ of the open analytic subspace $(\complement\, \overline{Y} \cup Y, \mathcal{O}_{\complement \overline{Y} \cup Y})$ of $(X, \mathcal{O}_X)$, provided $\complement\, \overline{Y} \cup Y$ is indeed open in $X$, i.e. $Y$ is locally closed in $X$.

*Examples.* The " single point " $(0, \mathbf{C})$ is an analytic subspace of the " double point " $(0, \mathbf{C}\{x\}/(x^2))$, but not conversely. The double point is, however, a closed analytic subspace of, e.g., $(\mathbf{C}, \mathcal{O}_{\mathbf{C}})$. A " point " of an analytic space will always mean a single point embedded in $(X, \mathcal{O}_X)$ by means of a map $(0, \mathbf{C}) \to (X, \mathcal{O}_X)$.

### *1.3. Operations on analytic spaces.*

In this section we shall write $X$ for the analytic space $(X, \mathcal{O}_X)$.

a) *Product.* By a general definition in the theory of categories, a product of two analytic spaces $X$, $X'$ is a triple $(Z, \pi, \pi')$ where $Z$ is an analytic space and $\pi : Z \to X$, $\pi' : Z \to X'$ are two morphisms with the following property:

Given any analytic space $Y$ and any pair $f: Y \to X$, $f' : Y \to X'$ of morphisms there exists a unique morphism $g : Y \to Z$ such that $f = \pi \circ g$, $f' = \pi' \circ g$.

For example, the product of $\mathbf{C}^p$ and $\mathbf{C}^q$ is $\mathbf{C}^{p+q}$, according to proposition 1.2.4.

We shall see that a product of analytic spaces always exists. The uniqueness of $g$ clearly implies the uniqueness of the product $(Z, \pi, \pi')$ up to isomorphism; we denote one such $Z$ by $X \times X'$.

To prove that the product always exists, let us suppose first that $X$ and $X'$ are special models, i.e. $X$ is defined by a triple $(U, f, F)$ where $U$ is open in $\mathbf{C}^n$, $F$ is a finite-dimensional complex linear space, and $f : U \to F$ is an analytic map; similarly for $X'$. We claim that the special model $Z$ defined by $(U \times U', f \times f', F \times F')$ is a product. Indeed, from the description of the morphisms into a special model provided by Proposition 1.2.5. it follows that we have natural maps $\pi : Z \to X$, $\pi' : Z \to X'$ induced by the projections $U \times U' \to U$, $U \times U' \to U'$. Also, if $f: Y \to X$ and $f' : Y \to X'$ are given, $g : Y \to Z$ is determined by

$$Y \begin{array}{l} \overset{f}{\nearrow} X \to U \searrow \\ \underset{f'}{\searrow} X' \to U' \nearrow \end{array} U \times U' .$$

In the general case we take $X \times X'$ as the ringed space whose topological underlying space in the cartesian product of the underlying space of $X$ and $X'$, and whose structure sheaf is given locally by the product of local models for $X$ and $X'$. (From the uniqueness "up to isomorphism" of the product results that these sheaves stick together in a well-determined way).

b) *Kernel of a double arrow.* If $X \underset{v}{\overset{u}{\rightrightarrows}} Y$ is a double arrow, i.e. a pair of morphisms, a kernel $X'$ of $(u, v)$ is an analytic subspace of $X$ such that the morphisms of an arbitrary analytic space $Z$ into $X'$ are exactly the morphisms $h$ of $Z$ into $X$ such that $u \circ h = v \circ h$. In other words, if $i : X' \to X$ is the natural map of $X'$ into $X$, the morphisms $h : Z \to X'$ satisfy $u \circ i \circ h = v \circ i \circ h$ and if a morphism $g : Z \to X$ satisfies $u \circ g = v \circ g$, then $g = i \circ h$ for some $h : Z \to X'$. To prove the existence of the kernel it suffices, again, to do this locally, i.e. for special models. If $X$ is defined by $(U, f, F)$ and $Y$ by $(V, g, G)$ we may (perhaps, after restricting $U$) extend $u$ and $v$ to maps $\bar{u}, \bar{v} : U \to E$ where $E$ denotes the complex linear space of which $V$ is an open subset. The kernel is then defined by the triple

$$\left(U, f \times (\bar{u} - \bar{v}), \quad F \times E\right).$$

It follows from the Proposition 1.2.5. that this special model satisfies the universal property of kernels.

*Example 1.* The kernel of $\mathbf{C} \underset{-t}{\overset{t}{\rightrightarrows}} \mathbf{C}$ is the simple point $\{0\}$, $t$ denoting the identity of $\mathbf{C}$.

*Example 2.* The kernel of $\mathbf{C} \underset{t+t^2}{\overset{t}{\rightrightarrows}} \mathbf{C}$ is $\{0\}$ counted as a double point.

c) *Fiber product.* If $u : X \to S$ and $v : Y \to S$ are given morphisms of analytic spaces, the fiber product $X \times_S Y$ of $X$ and $Y$ over $S$ is the kernel of the double arrow

$$X \times Y \underset{v \circ \pi'}{\overset{u \circ \pi}{\rightrightarrows}} S$$

where $\pi : X \times Y \to X$ and $\pi' : X \times Y \to Y$ are the maps defined by the product. Note that when $S$ is a simple point, $X \times_S Y = X \times Y$.

One may also introduce the category of analytic spaces over $S$. Its objects are morphisms $u : X \to S$ of an analytic space $X$ onto $S$ and its morphisms are morphisms $f : X \to Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {}_u\searrow \quad \swarrow_v & \\ & S & \end{array}$$

is commutative. The product in this category, i.e. the object satisfying the universal property given above for the product $X \times Y$, is then exactly the fiber product $X \times_S Y$. If $S$ is a point, we have the category of analytic spaces.

*Example 3.* If $U$ and $V$ are open subspaces of an analytic space $X$, the open subspace $U \cap V$ is isomorphic to $U \times_X V$. We may thus define, in general, the intersection of two analytic subspaces $X' \to X$ and $X'' \to X$ of $X$ to be the fiber product $X' \times_X X''$.

*Example 4.* If $\varphi : Y \to X$ is a morphism of analytic spaces and $a \in X$ a point, i.e. a map $a : (0, \mathbf{C}) \to X$ we may consider the space $Y(a) = Y \times_X a$. It is natural to call this the inverse image of $a$ under $\varphi$ and to denote it by $\varphi^{-1}(a)$; its underlying space is exactly $\varphi_0^{-1}(a)$.

If $\varphi_0(b) = a$, then $\mathcal{O}_{Y(a),b}$ is $\mathcal{O}_{Y,b}$ taken modulo the image under $\varphi^1 : \mathcal{O}_{X,a} \to \mathcal{O}_{Y,b}$ of the maximal ideal in $\mathcal{O}_{X,a}$.

*Example 5.* The pull-back of a linear bundle $E$ over $X$ by a map $Y \to X$ is exactly $Y \times_X E$.

### 1.4. *Relations between reduced and non-reduced spaces.*

We shall first characterize those analytic spaces which are reduced.

*Proposition 1.4.1.* A analytic space $(X, \mathcal{O}_X)$ is reduced if and only if $\mathcal{O}_{X,x}$ has no nilpotent element for $x$ arbitrary in $X$.

*Proof.* The necessity of the condition is obvious for $\mathcal{O}_X$ can be considered as a submodule of $\mathcal{C}_X$ if $(X, \mathcal{O}_X)$ is reduced.

Conversely, if $\mathcal{O}_{X,x}$ has no nilpotent elements, we shall prove that in any local model $(V, \mathcal{O}_V)$ for $(X, \mathcal{O}_X)$, a germ $g$ at $a \in V$ which vanishes on $V$ belongs to the ideal $\mathcal{I}$ defining $\mathcal{O}_V$. The Nullstellensatz implies that $g^k \in \mathcal{I}_a$ if $k$ is large enough. But it is then clear that $g \in \mathcal{I}_a$ if $\mathcal{O}_{V,a}/\mathcal{I}_a$ is free from nilpolent elements.

Given an analytic space $(X, \mathcal{O}_X)$ we can associate to it a reduced space in the following way. Let $\mathcal{N}_x$ be the ideal in $\mathcal{O}_{X,x}$ consisting of all nilpotent elements (the nil-radical of 0). Then $\mathcal{N} = U\mathcal{N}_x$ is a coherent sheaf by the Oka-Cartan theorem, for in a local model $(V, \mathcal{O}_V)$ for $(X, \mathcal{O}_X)$ we have $\mathcal{N}_X = (\mathcal{I}'/\mathcal{I})_X$ where $\mathcal{I}'$ is the sheaf of germs vanishing on $V$ and $\mathcal{I}$ the