

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 14 (1968)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ANALYTIC SPACES
Autor: Malgrange, Bernard
Kapitel: 4.2. Topology on (X, F) .
DOI: <https://doi.org/10.5169/seals-42341>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 14.03.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

If X is a Stein space, X_{red} is obviously also a Stein space. The converse is also true (see Grauert [2]).

Theorem 4.1.2. (“Theorems A and B ” of Cartan-Oka). Let F be an analytic coherent sheaf over a Stein space (X, \mathcal{O}_X) . Then

- 1) For any $x \in X$, $\Gamma(X, F)$ generates F_x over $\mathcal{O}_{X,x}$
- 2) For $p \geq 1$, one has $H^p(X, F) = 0$

This theorem will not be proved here (see f.i. [5] for the reduced case ; the general case is similar). We will need here only the following special case :

Let (X, \mathcal{O}_X) be a closed analytic subspace of a domain of holomorphy $U \subset \mathbb{C}^n$; if F is an analytic coherent sheaf on X , let \tilde{F} be the trivial extension of F to U ; then \tilde{F} is a coherent sheaf of \mathcal{O}_U modules, and theorems A and B are valid for \tilde{F} : therefore, they are true for F .

4.2. Topology on $\Gamma(X, F)$.

1. Let X be a closed analytic subspace of a domain of holomorphy $U \subset \mathbb{C}^n$; and, with the previous notations, suppose that \tilde{F} admits a *finite presentation* i.e. an exact sequence of sheaves of \mathcal{O}_U -modules

$$\mathcal{O}_U^q \xrightarrow{\alpha} \mathcal{O}_U^p \xrightarrow{\beta} \tilde{F} \rightarrow 0.$$

Applying theorem B to the exact sequences

$$0 \rightarrow \text{Im } \alpha \rightarrow \mathcal{O}_U^p \rightarrow \tilde{F} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \text{Ker } \alpha \rightarrow \mathcal{O}_U^q \rightarrow \text{Im } \alpha \rightarrow 0$$

we get an exact sequence

$$\Gamma(U, \mathcal{O}_U)^q \xrightarrow{\Gamma(U, \alpha)} \Gamma(U, \mathcal{O}_U)^p \xrightarrow{\Gamma(U, \beta)} \Gamma(U, \tilde{F}) \rightarrow 0.$$

The space $\Gamma(U, \mathcal{O}_U)$, with the topology of uniform convergence on compact sets is a Frechet space. And we claim that, for that topology, $\text{Im } \Gamma(U, \alpha)$ is closed. For, if f is adherent to $\text{Im } \Gamma(U, \alpha)$, it results easily from Krull's theorem (see Appendix) that, for $x \in U$, we have $f_x \in \text{Im } (\alpha_x)$, hence $f \in \Gamma(U, \text{Im } \alpha)$; but, according to theorem B , the mapping $\Gamma(U, \mathcal{O}_U)^q \rightarrow \Gamma(U, \text{Im } \alpha)$ is surjective.

Now, with the quotient topology, $\Gamma(X, F) \simeq \Gamma(U, \tilde{F}) \simeq \Gamma(U, \mathcal{O}_U) / \text{Im } \Gamma(U, \alpha)$ is a Frechet space. This topology does not depend on the given presentation of \tilde{F} (in fact, it does not even depend on the imbedding $X \rightarrow U$, but we shall not need it here). For, suppose we have a second presentation

$$\Gamma(U, \mathcal{O}_U)^{q'} \xrightarrow{\alpha'} \Gamma(U, \mathcal{O}_U)^{p'} \xrightarrow{\beta'} \tilde{F} \rightarrow 0.$$

As $\Gamma(U, \mathcal{O}_U)^p$ is free over $\Gamma(U, \mathcal{O}_U)$, we can find a $\Gamma(U, \mathcal{O}_U)$ -linear map $\Gamma(U, \mathcal{O}_U)^p \xrightarrow{\gamma} \Gamma(U, \mathcal{O}_U)^{p'}$ such that $\beta = \beta' \circ \gamma$; this induces a continuous map

$$\Gamma(U, \mathcal{O}_U)^p / \text{Im } \Gamma(U, \alpha) \rightarrow \Gamma(U, \mathcal{O}_U)^{p'} / \text{Im } \Gamma(U, \alpha')$$

which is bijective, hence bicontinuous according to the closed graph theorem.

2. General case

If X is an analytic space and F an analytic coherent sheaf on X , we can find a) a locally finite covering of X by open subspaces X_i , b) for each i , a morphism $X_i \rightarrow U_i$, U_i open polycylinder in \mathbf{C}^{n_i} , which identifies X_i with a closed subspace of U_i c) for each i , a coherent sheaf \tilde{F}_i on U_i admitting a finite presentation, such that \tilde{F}_i is the extension of $F|_{X_i}$.

On $\Gamma(X_i, F|_{X_i})$ we have already defined a topology; further, consider the natural injection

$$\Gamma(X, F) \rightarrow \prod_i \Gamma(X_i, F|_{X_i})$$

We claim that its image is closed. For, (f_i) belongs to the image if and only if, for all $x \in X_i \cap X_j (= X_i \times_X X_j)$, we have $(f_i)_x = (f_j)_x$; and the fact that this relations define a closed subspace results easily from Krull's theorem.

This gives a topology of Frechet space on $\Gamma(X, F)$. It does not depend on the chosen covering (if one has two coverings, one considers a common refinement, and one applies again Krull's theorem and the closed graph theorem; we leave the details to the reader). One proves in the same way that if X' is an open subspace of X , the restriction map $\Gamma(X, F) \rightarrow \Gamma(X', F|_{X'})$ is continuous. If X' is relatively compact in X , then the restriction map is compact (this can be seen by choosing a covering X'_j of X' of the same type, such that, for any j , there exist i with $X'_j \subset X_i$, X'_j relatively compact in X_i , and applying Ascoli's theorem).

4.3. Topology on $H^p(X, F)$

We consider a locally finite covering $\mathcal{U} = \{X_i\}_{i \in I}$ by open subspaces of the preceding type. If we have $i_0, \dots, i_p \in I$, we consider the natural morphisms

$$X_{i_0 \dots i_p} = X_{i_0} \times_X \dots \times_X X_{i_p} \rightarrow X_{i_0} \times \dots \times X_{i_p} \rightarrow U_{i_0} \times \dots \times U_{i_p}$$