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If X is a Stein space, X_{red} is obviously also a Stein space. The converse is also true (see Grauert [2]).

Theorem 4.1.2. (" Theorems A and B " of Cartan-Oka). Let F be an analytic coherent sheaf over a Stein space $(X, \mathcal{O}_X)$. Then

1) For any $x \in X$, $\Gamma(X, F)$ generates F_x over $\mathcal{O}_{X,x}$
2) For $p \geqslant 1$, one has $H^p(X, F) = 0$

This theorem will not be proved here (see f.i. [5] for the reduced case ; the general case is similar). We will need here only the following special case :

Let $(X, \mathcal{O}_X)$ be a closed analytic subspace of a domain of holomorphy $U \subset \mathbf{C}^n$; if F is an analytic coherent sheaf on X, let $\tilde{F}$ be the trivial extension of F to U ; then $\tilde{F}$ is a coherent sheaf of $\mathcal{O}_U$ modules, and theorems A and B are valid for $\tilde{F}$: therefore, they are true for F.

4.2. *Topology on $\Gamma(X, F)$.*

1. Let X be a closed analytic subspace of a domain of holomorphy $U \subset \mathbf{C}^n$; and, with the previous notations, suppose that $\tilde{F}$ admits a *finite presentation* i.e. an exact sequence of sheaves of $\mathcal{O}_U$-modules

$$\mathcal{O}_U^q \xrightarrow{\alpha} \mathcal{O}_U^p \xrightarrow{\beta} \tilde{F} \to 0 .$$

Applying theorem B to the exact sequences

$$0 \to \operatorname{Im} \alpha \to \mathcal{O}_U^p \to \tilde{F} \to 0 \quad \text{and} \quad 0 \to \operatorname{Ker} \alpha \to \mathcal{O}_U^q \to \operatorname{Im} \alpha \to 0$$

we get an exact sequence

$$\Gamma(U, \mathcal{O}_U)^q \xrightarrow{\Gamma(U,\alpha)} \Gamma(U, \mathcal{O}_U)^p \xrightarrow{\Gamma(U,\beta)} \Gamma(U, \tilde{F}) \to 0 .$$

The space $\Gamma(U, \mathcal{O}_U)$, with the topology of uniform convergence on compact sets is a Frechet space. And we claim that, for that topology, $\operatorname{Im} \Gamma(U, \alpha)$ is closed. For, if f is adherent to $\operatorname{Im} \Gamma(U, \alpha)$, it results easily from Krull's theorem (see Appendix) that, for $x \in U$, we have $f_x \in \operatorname{Im}(\alpha_x)$, hence $f \in \Gamma(U, \operatorname{Im} \alpha)$; but, according to theorem B, the mapping $\Gamma(U, \mathcal{O}_U)^q \to \Gamma(U, \operatorname{Im} \alpha)$ is surjective.

Now, with the quotient topology, $\Gamma(X, F) \simeq \Gamma(U, \tilde{F}) \simeq \Gamma(U, \mathcal{O}_U) / \operatorname{Im} \Gamma(U, \alpha)$ is a Frechet space. This topology does not depend on the given presentation of $\tilde{F}$ (in fact, it does not even depend on the imbedding $X \to U$, but we shall not need it here). For, suppose we have a second presentation

$$\Gamma(U,\mathcal{O}_U)^{q'} \xrightarrow{\alpha'} \Gamma(U,\mathcal{O}_U)^{p'} \xrightarrow{\beta'} \tilde{\tilde{F}} \to 0 .$$

As $\Gamma(U, \mathcal{O}_U)^p$ is free over $\Gamma(U, \mathcal{O}_U)$, we can find a $\Gamma(U, \mathcal{O}_U)$-linear map $\Gamma(U, \mathcal{O}_U)^p \xrightarrow{\gamma} \Gamma(U, \mathcal{O}_U)^{p'}$ such that $\beta=\beta' \circ \gamma$; this induces a continuous map

$$\Gamma(U,\mathcal{O}_U)^p/\mathrm{Im}\ \Gamma(U,\alpha) \to \Gamma(U,\mathcal{O}_U)^{p'}/\mathrm{Im}\ \Gamma(U,\alpha')$$

which is bijective, hence bicontinuous according to the closed graph theorem.

2. *General case*

If X is an analytic space and F an analytic coherent sheaf on X, we can find a) a locally finite covering of X by open subspaces X_i, b) for each i, a morphism $X_i \to U_i$, U_i open polycylinder in $\mathbf{C}^{n_i}$, which identifies X_i with a closed subspace of U_i c) for each i, a coherent sheaf $\tilde{F}_i$ on U_i admitting a finite presentation, such that $\tilde{F}_i$ is the extension of $F/_{X_i}$.

On $\Gamma(X_i, F\mid_{X_i})$ we have already defined a topology ; further, consider the natural injection

$$\Gamma(X,F) \to \prod_i \Gamma(X_i, F|_{X_i}))$$

We claim that its image is closed. For, (f_i) belongs to the image if and only if, for all $x \in X_i \cap X_j$ ($= X_i \times_X X_j$), we have $(f_i)_x = (f_j)_x$; and the fact that this relations define a closed subspace results easily from Krull's theorem.

This gives a topology of Frechet space on $\Gamma(X, F)$. It does not depend on the chosen covering (if one has tavo coverings, one considers a common refinement, and one applies again Krull's theorem and the closed graph theorem; we leave the details to the reader). One proves in the same way that if X' is an open subspace of X, the restriction map $\Gamma(X, F) \to \Gamma(X', F\mid_{X'})$ is continuous. If X' is relatively compact in X, then the restriction map is compact (this can be seen by choosing a covering X'_j of X' of the same type, such that, for any j, there exist i with $X'_j \subset X_i$, X'_j relatively compact in X_i, and applying Ascoli's theorem).

4.3. *Topology on $H^p(X, F)$*

We consider a locally finite covering $\mathcal{U} = \{ X_i \}_{i\in I}$ by open subspaces of the preceding type. If we have $i_0, \ldots, i_p \in$ I, we consider the natural morphisms

$$X_{i_0\ldots i_p} = X_{i_0} \underset{X}{\times} \ldots \underset{X}{\times} X_{i_p} \to X_{i_0} \times \ldots \times X_{i_p} \to U_{i_0} \times \ldots \times U_{i_p}$$