

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 14 (1968)  
**Heft:** 1: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ANALYTIC SPACES  
**Autor:** Malgrange, Bernard  
**Kapitel:** 4.3. Topology on  $H^p(X, F)$   
**DOI:** <https://doi.org/10.5169/seals-42341>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

**Download PDF:** 13.03.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

$$\Gamma(U, \mathcal{O}_U)^{q'} \xrightarrow{\alpha'} \Gamma(U, \mathcal{O}_U)^{p'} \xrightarrow{\beta'} \tilde{F} \rightarrow 0.$$

As  $\Gamma(U, \mathcal{O}_U)^p$  is free over  $\Gamma(U, \mathcal{O}_U)$ , we can find a  $\Gamma(U, \mathcal{O}_U)$ -linear map  $\Gamma(U, \mathcal{O}_U)^p \xrightarrow{\gamma} \Gamma(U, \mathcal{O}_U)^{p'}$  such that  $\beta = \beta' \circ \gamma$ ; this induces a continuous map

$$\Gamma(U, \mathcal{O}_U)^p / \text{Im } \Gamma(U, \alpha) \rightarrow \Gamma(U, \mathcal{O}_U)^{p'} / \text{Im } \Gamma(U, \alpha')$$

which is bijective, hence bicontinuous according to the closed graph theorem.

## 2. General case

If  $X$  is an analytic space and  $F$  an analytic coherent sheaf on  $X$ , we can find a) a locally finite covering of  $X$  by open subspaces  $X_i$ , b) for each  $i$ , a morphism  $X_i \rightarrow U_i$ ,  $U_i$  open polycylinder in  $\mathbf{C}^{n_i}$ , which identifies  $X_i$  with a closed subspace of  $U_i$  c) for each  $i$ , a coherent sheaf  $\tilde{F}_i$  on  $U_i$  admitting a finite presentation, such that  $\tilde{F}_i$  is the extension of  $F|_{X_i}$ .

On  $\Gamma(X_i, F|_{X_i})$  we have already defined a topology; further, consider the natural injection

$$\Gamma(X, F) \rightarrow \prod_i \Gamma(X_i, F|_{X_i})$$

We claim that its image is closed. For,  $(f_i)$  belongs to the image if and only if, for all  $x \in X_i \cap X_j (= X_i \times_x X_j)$ , we have  $(f_i)_x = (f_j)_x$ ; and the fact that this relations define a closed subspace results easily from Krull's theorem.

This gives a topology of Frechet space on  $\Gamma(X, F)$ . It does not depend on the chosen covering (if one has two coverings, one considers a common refinement, and one applies again Krull's theorem and the closed graph theorem; we leave the details to the reader). One proves in the same way that if  $X'$  is an open subspace of  $X$ , the restriction map  $\Gamma(X, F) \rightarrow \Gamma(X', F|_{X'})$  is continuous. If  $X'$  is relatively compact in  $X$ , then the restriction map is compact (this can be seen by choosing a covering  $X'_j$  of  $X'$  of the same type, such that, for any  $j$ , there exist  $i$  with  $X'_j \subset X_i$ ,  $X'_j$  relatively compact in  $X_i$ , and applying Ascoli's theorem).

### 4.3. Topology on $H^p(X, F)$

We consider a locally finite covering  $\mathcal{U} = \{X_i\}_{i \in I}$  by open subspaces of the preceding type. If we have  $i_0, \dots, i_p \in I$ , we consider the natural morphisms

$$X_{i_0 \dots i_p} = X_{i_0} \times_X \dots \times_X X_{i_p} \rightarrow X_{i_0} \times \dots \times X_{i_p} \rightarrow U_{i_0} \times \dots \times U_{i_p}$$

which makes  $X_{i_0}, \dots, i_p$  isomorphic with a closed subspace of  $U_{i_0} \times \dots \times U_{i_p}$  (the hypothesis that  $X$  is separated is essential here! See remark at the end of this paragraph), therefore,  $X_{i_0}, \dots, i_p$  satisfies theorems  $A$  and  $B$ ; more generally, if a finite number of open subspaces of  $X$  is Stein, their intersection is also Stein.

Introduce a total order on  $I$ . Given an analytic coherent sheaf on  $X$ , we can identify the alternating cochains of degree  $p$  of the covering  $\mathcal{U}$  with values in  $F$  with the space

$$C^p(\mathcal{U}, F) = \prod_{i_0 < i_1 < \dots < i_p} \Gamma(X_{i_0 \dots i_p}, F|_{X_{i_0 \dots i_p}}).$$

This is a Frechet space, and the differential  $d: C^p(\mathcal{U}, F) \rightarrow C^{p+1}(\mathcal{U}, F)$  is clearly continuous. Therefore the kernel  $Z^p(\mathcal{U}, F)$  is a closed subspace of  $C^p(\mathcal{U}, F)$ . We denote  $B^p(\mathcal{U}, F)$  the image of  $C^{p-1}(\mathcal{U}, F)$  under  $d$ , and we consider on  $H^p(\mathcal{U}, F) = Z^p(\mathcal{U}, F)/B^p(\mathcal{U}, F)$  the quotient topology; according to Leray's theorem, there is a natural isomorphism  $H^p(X, F) \simeq H^p(\mathcal{U}, F)$ .

This gives a topology on  $H^p(X, F)$  of a quotient of a Frechet space. In general, this topology is *not separated*.

We prove now that this topology is independent of the covering  $\mathcal{U}$ ; to do that, it is sufficient to consider a refinement  $\mathcal{U}' = \{X'_j\}_{j \in J}$  of  $\mathcal{U}$  of the same type, a map  $\varphi: J \rightarrow I$  such that  $X'_j \subset X_{\varphi(j)}$  for any  $j$  to consider the map defined by  $\varphi: C^*(\mathcal{U}, F) = \bigoplus_p C^p(\mathcal{U}, F) \xrightarrow{\varphi} C^*(\mathcal{U}', F)$  and to prove that the induced map  $\bar{\rho}: H^p(\mathcal{U}, F) \rightarrow H^p(\mathcal{U}', F)$  is an isomorphism.

First,  $\bar{\rho}$  is obviously continuous and bijective; so, according to the closed graph theorem, all that we have to prove is that  $\bar{\rho}$  maps the adherence of 0 onto the adherence of zero; to do that, we consider  $\bar{a}' \in H^p(\mathcal{U}, F)$ , which is adherent to zero; this means that  $\bar{a}'$  is the class modulo  $B^p(\mathcal{U}', F)$  of some  $a' \in Z^p(\mathcal{U}', F)$  which is adherent to  $B^p(\mathcal{U}', F)$ ; therefore, we have

$$a' = \lim_{n \rightarrow \infty} db'_n, \quad b'_n \in C^{p-1}(\mathcal{U}', F).$$

Now, the map

$$Z^p(\mathcal{U}, F) \oplus C^{p-1}(\mathcal{U}', F) \xrightarrow{(\rho, d)} Z^p(\mathcal{U}', F)$$

is surjective hence, according to the closed graph theorem, we can find converging sequences  $a_n \in Z^p(\mathcal{U}, F)$  and  $b''_n \in C^{p-1}(\mathcal{U}', F)$  such that  $db''_n = \rho(a_n) + db''_n$ ; but,  $\bar{\rho}$  being an isomorphism, we have  $a_n = d\alpha_n$ ,  $\alpha_n \in C^{n-1}(\mathcal{U}, F)$ ; if we put  $b = \lim_{n \rightarrow \infty} b''_n$ ,  $a = \lim_{n \rightarrow \infty} a_n$ , we find that  $a \in B^p(\mathcal{U}, F)$  and that the class  $a$  of  $a$  in  $H^p(\mathcal{U}, F)$  verifies  $\bar{\rho}(a) = \bar{a}'$ ; this proves the result.

*Remark.* If  $X$  is not separated, an intersection of two open Stein subspaces of  $X$  need not be Stein; take f.i. for  $X$  two copies of  $\mathbb{C}^2$ , identified everywhere except at  $O$ ; there is an obvious covering of  $X$  by two open subspaces, identicals with  $\mathbb{C}^2$ ; but their intersection is  $\mathbb{C}^2 - \{O\}$ , and therefore is not Stein!

#### 4.4. The finiteness theorem

*Theorem 4.4.1.* (Cartan — Serre). Let  $X$  be a compact analytic space, and  $F$  be a coherent analytic sheaf on  $X$ . Then, for every  $p \geq 0$   $H^p(X, F)$  is separated and finite dimensional.

We shall give two proofs of this theorem ; both are interesting for further applications.

*1st proof.* Let  $\{X_i\}$  and  $\{X'_i\}$  be two finite coverings of  $X$  of the type considered in the previous articles, such that, for every  $i$ ,  $X'_i$  is relatively compact in  $X_i$ . Then, if we denote by  $\mathcal{U}$  (resp.  $\mathcal{U}'$ ) the covering  $\{X_i\}$  (resp.  $\{X'_i\}$ ), the natural restriction map  $C^p(\mathcal{U}, F) \rightarrow C^p(\mathcal{U}', F)$  is compact.

Consider now the map

$$(\rho, d) : Z^p(\mathcal{U}, F) \oplus C^{p-1}(\mathcal{U}', F) \rightarrow Z^p(\mathcal{U}', F)$$

this map is surjective, and we have  $(, d\rho) = (\rho, 0) + (0, d)$ ,  $(\rho, 0)$  being compact ; then the following lemma proves that  $\text{Im}(0, d)$  is closed and finite codimensional, q.e.d.

*Lemma 4.4.2.* Let  $E$  and  $F$  two Frechet spaces,  $u_1$  and  $u_2$  two linear continuous maps  $E \rightarrow F$  such that  $u_1 + u_2$  is surjective, and  $u_1$  compact. Then  $\text{Im}(u_2)$  is closed and finite codimensional. For the proof, see e.g. [5].

*2nd proof.* Consider  $\mathcal{U}$  and  $\mathcal{U}'$  as above, and consider the map  $(\rho, d) : C^{p-1}(\mathcal{U}, F)/Z^{p-1}(\mathcal{U}, F) \rightarrow [C^{p-1}(\mathcal{U}', F)/Z^{p-1}(\mathcal{U}', F)] \oplus Z^p(\mathcal{U}, F)$   $(\rho, d)$  is clearly injective. I claim that its image is closed: In fact, since  $\bar{\rho} : H^p(\mathcal{U}, F) \rightarrow H^p(\mathcal{U}', F)$  is injective, this image consists of the pairs  $(\bar{a}', b)$ ,  $a' \in C^{p-1}(\mathcal{U}', F)$ ,  $b \in Z^p(\mathcal{U}, F)$  such that  $da' = \rho b$ , which proves the assertion.

Now we have  $(\rho, d) = (\rho, 0) + (0, d)$  and  $(\rho, 0)$  is compact. By a well-known lemma, it results that  $\text{Im}(0, d)$  is closed, which means that  $H^p(\mathcal{U}, F)$  is separated.

Finally, since  $\bar{\rho}$  is compact, and is an isomorphism, it follows that the identity map of  $H^p(\mathcal{U}, F)$  into itself is compact ; therefore this space is finite dimensional ; this proves the theorem.