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MEROMORPHIC MAPPINGS

by K. Stein

Introduction

We study meromorphic mappings of complex spaces. The notion of meromorphic mapping we use was introduced by Remmert [9], [11][1]. Some part of the material dealt with in these lectures is contained in [15] and we shall therefore not give proofs for all statements.

The first sections are preliminary. The concept of correspondence is discussed and used to define meromorphic mappings (these are not mappings in the usual sense). Extension problems are studied in Section 4. Essential use is made of the extension theorem for analytic sets first proved by Thullen [21] in a special case and later generalized by Remmert and Stein [13]. The final section deals with maximal meromorphic mappings.

1. Correspondences

Let X and Y be sets. A *correspondence*, denoted $f : X \underset{k}{\rightarrow} Y$, assigns to each $x \in X$ a subset $f(x) \subset Y$, which may be empty. $f : X \underset{k}{\rightarrow} Y$ is called empty if $f(x) = \varnothing$ for all $x \in X$. For $A \subset X$ we set $f(A) = \bigcup_{x \in A} f(x)$. A mapping $\varphi : X \rightarrow Y$ is looked upon as a special correspondence (we do not distinguish between a set consisting of one element and the element).

Each correspondence $f : X \underset{k}{\rightarrow} Y$ can be characterized by its *graph* $G_f = \{ (x, y) \mid x \in X, y \in f(x) \} \subset X \times Y$. The projection maps of G_f into X and Y are denoted by $\check{f} : G_f \rightarrow X$ and $\hat{f} : G_f \rightarrow Y$. Then, we have $f(x) = \hat{f}\left(\check{f}^{-1}(x)\right)$. If $f : X \underset{k}{\rightarrow} Y$, $f' : X \underset{k}{\rightarrow} Y$ are correspondences, we say that f is contained in f' if $G_f \subset G_{f'}$. For a subset $A \subset X$ we define the *restriction*

[1]) Another notion of meromorphic mapping and related concepts were defined by W. Stoll [16], [17].

$f|A : A \underset{k}{\to} Y$ by setting $G_{f|A} = G_f \cap (A \times Y)$. To every correspondence $f : X \underset{k}{\to} Y$ there is associated the *inverse correspondence* $f^{-1} : Y \underset{k}{\to} X$ whose graph is $G_{f^{-1}} = \{(y, x) \mid (x, y) \in G_f\}$. We have the rule $(f^{-1})^{-1} = f$. The *Cartesian product* $f \times f_1 : X \times X_1 \underset{k}{\to} Y \times Y_1$ of two correspondences $f: X \underset{k}{\to} Y$ and $f_1 : X_1 \underset{k}{\to} Y_1$ assigns, by definition, to $(x, x_1) \in X \times X_1$ the set $f(x) \times f_1(x_1)$. If $X = X_1$, we define the *junction* $(f, f_1) : X \underset{k}{\to} Y \times Y_1$ by $(f, f_1)(x) = f(x) \times f_1(x)$. The *product* $g \circ f: X \underset{k}{\to} Z$ of the correspondences $f : X \underset{k}{\to} Y$ and $g : Y \underset{k}{\to} Z$ is defined by $g \circ f(x) = g(f(x))$. Hence, $(f, f_1) = (f \times f_1) \circ (I_X, I_X)$ where I_X is the identity mapping of X. We have the following rules: $(f \times f_1)^{-1} = f^{-1} \times f_1{}^{-1}$, $h \circ (g \circ f) = (h \circ g) \circ f$, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Definition 1. Let X and Y be topological spaces. A correspondence $f: X \underset{k}{\to} Y$ is *continuous* at $x \in X$ if

1) $f(x)$ is quasicompact, and

2) given a neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$.

The correspondence f is *continuous* if it is continuous at every $x \in X$.[1]

Proposition 1. Let $f : X \underset{k}{\to} Y$ be a correspondence such that $f(x)$ is quasicompact for all $x \in X$. Then f is continuous if and only if f^{-1} is closed (in the sense that the images of closed sets are closed).

Proof. Let f be continuous and let N be a closed set in Y. Assume that $f^{-1}(N)$ is not closed, then there is a point $x \in \overline{f^{-1}(N)} \cap (X - f^{-1}(N))$. We have $f(x) \in Y - N$ since $x \in X - f^{-1}(N)$, hence $Y - N$ is a neighborhood of $f(x)$. Because of the continuity of f there exists a neighborhood U of x such that $f(U) \subset Y - N$. It follows $U \cap f^{-1}(N) = \varnothing$, but this contradicts the assumption that $x \in \overline{f^{-1}(N)}$. Assume now that f^{-1} is closed. Let x be a point of X and V an open neighborhood of $f(x)$. Then $f^{-1}(Y - V)$ is closed and does not contain x, therefore $U = X - f^{-1}(Y - V)$ is a neighborhood of x, and we have $f(U) \subset V$. Hence f is continuous.

Remark. The statement of Proposition 1 becomes false if " closed " is replaced by " open " as can be seen by simple examples.

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1) This definition and some of the following developments are due to K. Wolffhardt [22].

Proposition 2. If $f: X \underset{k}{\rightarrow} Y$ is a continuous correspondence, $f(K)$ is quasicompact for every quasicompact set $K \subset X$.

Proof. Consider a covering of $f(K)$ by open sets V_i. For each $x \in K$ there are finitely many V_i which cover $f(x)$, let $\tilde{V}_x$ be the union of those V_i. f is continuous at x, hence there is a neighborhood U_x of x such that $f(U_x) \subset \tilde{V}_x$. Now finitely many U_x, say $U_{x_1}, \ldots, U_{x_n}$, cover K, hence $\tilde{V}_{x_1} \cup \ldots \cup \tilde{V}_{x_n} \supset f(K)$, therefore finitely many V_i cover $f(K)$.

Proposition 3. Let $f: X \underset{k}{\rightarrow} Y$, $f_1: X_1 \underset{k}{\rightarrow} Y_1$, $f_1': X \underset{k}{\rightarrow} Y_1$, $g: Y \underset{k}{\rightarrow} Z$ be continuous correspondences. Then $f \times f_1$, (f, f_1'), $g \circ f$ are continuous.

Proof. For $g \circ f$, the assertion follows by applying propositions 1 and 2, Furthermore, $f \times f_1(x, x_1) = f(x) \times f_1(x_1)$ is quasicompact for all $x \in X$. $x_1 \in X_1$ since $f(x)$ and $f_1(x_1)$ are quasicompact. Let V be a neighborhood of $f(x) \times f_1(x_1)$; V contains a neighborhood $W \times W_1$ of $f(x) \times f_1(x_1)$ where W, W_1 are neighborhoods of $f(x)$ resp. $f_1(x_1)$. There are neighborhoods U, U_1 of x resp. x_1 such that $f(U) \subset W$, $f_1(U_1) \subset W_1$, then $f \times f_1(U \times U_1) \subset$ $\subset W \times W_1$; hence $f \times f_1$ is continuous. As for (f, f_1') one has $(f, f_1') =$ $(f \times f_1') \circ (I_X, I_X)$, therefore (f, f_1') is continuous because $f \times f_1'$ and (I_X, I_X) are continuous.

Proposition 4. A correspondence $f: X \underset{k}{\rightarrow} Y$ is continuous if and only if $\check{f}^{-1}: X \underset{k}{\rightarrow} G_f$ is continuous.

Proof. Since $f = \hat{f} \circ \check{f}^{-1}$, the continuity of $\check{f}^{-1}$ implies that of f by proposition 3. Let f be continuous and let x be a point of X. Since $\check{f}^{-1}(x)$ is homeomorphic to $f(x)$, it is quasicompact. Let $W \supset \check{f}^{-1}(x)$ be open in G_f. We can cover $\check{f}^{-1}(x)$ by a finite number of sets of the form $(U_i \times V_i) \cap$ $\cap G_f \subset W$, $U_i \ni x$ open in X and V_i open in Y. Then $V = \cup V_i \supset f(x)$ and there exists a neighborhood U' of x such that $f(U') \subset V$. If $U = (\cap U_i) \cap U'$, $\check{f}^{-1}(U) \subset W$.

It follows that a correspondence f is continuous if and only if the projection $\check{f}$ is a proper map, that is, $\check{f}$ is continuous, closed, and $\check{f}^{-1}(x)$ is always quasicompact.

Proposition 5. If $f: X \underset{k}{\rightarrow} Y$ is continuous and Y is a Hausdorff space, G_f is closed in $X \times Y$.

Definition 2. A correspondence f is *proper* if f and f^{-1} are continuous.[1]

Proposition 6. If $f: X \underset{k}{\rightarrow} Y$, $f_1: X_1 \underset{k}{\rightarrow} Y_1$, $g: Y \underset{k}{\rightarrow} Z$ are proper, then $f \times f_1$ and $g \circ f$ are proper.

The junction of two proper correspondences need not, however, be proper. The diagonal mapping (I_X, I_X) serves as an example if X is not a Hausdorff space. If X is Hausdorff, the junction (f, f') of proper correspondences $f: X \underset{k}{\rightarrow} Y$ and $f': X \underset{k}{\rightarrow} Y'$ remains proper.

Proposition 7. Let $f: X \underset{k}{\rightarrow} Y$, $f_1: X \underset{k}{\rightarrow} Y_1$, $g: Y \underset{k}{\rightarrow} Z$ be continuous where all the spaces are locally compact. Then we have:

1) If f is proper, then (f, f_1) and (f_1, f) are proper,

2) If $g \circ f$ is proper and g^{-1} surjective, then f is proper,

3) If $g \circ f$ is proper and f surjective, then g is proper.

2. Holomorphic correspondences

We consider reduced complex spaces (X, θ) where X is assumed Hausdorff and where the structure sheaf θ has no nilpotent elements. For the definition and related concepts we refer to [8]. The structure sheaf is usually omitted in the notation.

Definition 3. Let X and Y be complex spaces. A correspondence $f: X \underset{k}{\rightarrow} Y$ is called *holomorphic* if

1) f is continuous,

2) the graph G_f is an analytic set in $X \times Y$.

If only the condition 2) is fulfilled, f is said to be *weakly holomorphic.*

Let $f: X \underset{k}{\rightarrow} Y$ be weakly holomorphic. Then f^{-1} is weakly holomorphic; furthermore, if $A \subset X$ is analytic in X, $f \mid A$ is weakly holomorphic. Since $\check{f}^{-1}(x) = G_f \cap (\{x\} \times Y)$, $x \in X$, is analytic in G_f, $f(x) = \hat{f}(\check{f}^{-1}(x))$

1) Compare [3] where another notion of proper correspondence is defined.