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FLATNESS AND PRIVILEGE

by A. DOUADY

I. FLAT MORPHISMS

§ 1. Analytic subspaces of an analytic space

Let Y_1 and Y_2 be closed analytic subspaces of an analytic space X , and let them be defined by the \mathcal{O}_X ideals J_1, J_2 .

Definition 1: We say that Y_1 is *analytically included* in Y_2 , and we write $Y_1 \subset Y_2$, when $J_1 \supset J_2$.

Remark: The analytic inclusion implies the set theoretic inclusion, but the converse is not true.

Example: $X = (\mathbf{C}, \mathcal{O}_{\mathbf{C}})$; $J_1 = (x)$, $J_2 = (x^2)$. The space Y_1 is a simple point, Y_2 is a double point, $Y_1 \not\subset Y_2$, while they have the same underlying set.

Definition 2: The subspace $Y_1 \cup Y_2$ is the smallest subspace of X containing Y_1 and Y_2 , and it is defined by $J_1 \cap J_2$. The subspace $Y_1 \cap Y_2$ is the biggest subspace of X contained in both Y_1 and Y_2 , and it is defined by $J_1 + J_2$.

Remark: The underlying set of $Y_1 \cup Y_2$ (Resp. $Y_1 \cap Y_2$) is the union (Resp. intersection) of the underlying sets of Y_1 and Y_2 . However \cup and \cap of analytic spaces do not satisfy the distributivity laws which hold in set-theory: $(Y_1 \cup Y_2) \cap Y_3$ contains $Y_1 \cap Y_3$ and $Y_2 \cap Y_3$, and therefore their union; similarly $(Y_1 \cap Y_2) \cup Y_3 \subset (Y_1 \cup Y_3) \cap (Y_2 \cup Y_3)$. In general the converse inclusions do not hold.

Example: Let $X = \mathbf{C}^2$ and Y_1, Y_2, Z be given by ideals $(x-y)$, $(x+y)$ and (x) respectively.

$(Y_1 \cup Y_2) \cap Z$ is $\{0\}$ provided with $\mathbf{C}\{y\}/(y^2)$, while $(Y_1 \cap Y_2) \cup (Y_2 \cap Z)$ is the reduced space $\{0\}$. On the other hand: $Y_1 \cap Y_2 \subset Z$, $(Y_1 \cap Y_2) \cup Z = Z$, while $(Y_1 \cup Z) \cap (Y_2 \cup Z)$ is the space defined by the ideal (x^2, xy) . Its local ring at the origin is $\mathbf{C}\{x, y\}/(x^2, xy)$ in which x is nilpotent.

Definition 3 : Let X', X be analytic spaces, Y a closed analytic subspace of X defined by J , and $f = (f_0, f^1) : X' \rightarrow X$ a morphism.

The inverse image of Y by $f, f^{-1}(Y)$, is the analytic subspace Y' of X' defined by the ideal $J' = f^1(J) \mathcal{O}_{X'}$.

The inverse image of a simple point x in X is called the f -fiber over x , and is denoted by $f^{-1}(x)$ or $X'(x)$.

Proposition 1 : If $f = (f_0, f^1) : X' \rightarrow X$ is a morphism of analytic spaces, and Y is a subspace of X , then $f^{-1}(Y) \simeq \underset{X}{Y \times X'}$.

Proof : Let T be any analytic space, and $g : T \rightarrow X'$ a morphism. Then g can be considered as a morphism from T to $f^{-1}(Y)$ if and only if $f \circ g$ can be considered as a morphism from T to Y . Thus $f^{-1}(Y)$ and $\underset{X}{X' \times X}$ are solutions of the same universal problem.

§ 2. Analytic pull-back

In the following we want to generalize the notion of inverse image of a subspace.

We shall first recall the basic properties of the tensor product $E \otimes_A F$, where A is a commutative ring and E, F are two A -modules.

(1°) $E \otimes A^n = E^n \quad (n \in \mathbb{N})$

(2°) If the sequence of A -modules $F' \rightarrow F \rightarrow F'' \rightarrow 0$ is exact, then also the sequence $E \otimes F' \rightarrow E \otimes F \rightarrow E \otimes F'' \rightarrow 0$ is exact. (Right exactness of the tensor product)

(3°) If $(F_i)_{i \in I}; f_{ij} : F_j \rightarrow F_i$ is an inductive system, then

$$E \otimes \lim_{\rightarrow} F_i = \lim_{\rightarrow} (E \otimes F_i).$$

On the other hand these properties characterize completely the functor \otimes .

Definition 1 : Let $f = (f_0, f^1) : X' \rightarrow X$ be a morphism of analytic spaces, and \mathcal{E} an \mathcal{O}_X -module. Then $f_0^* \mathcal{E}$ is an $f_0^* \mathcal{O}_X$ -module and $\mathcal{O}_{X'}$ is also an $f_0^* \mathcal{O}_X$ -module (by $f^1 : f_0^* \mathcal{O}_X \rightarrow \mathcal{O}_{X'}$).

The analytic pull-back $f^* \mathcal{E}$ of \mathcal{E} by f is defined by scalar extension:

$$f^* \mathcal{E} = f_0^* \mathcal{E} \otimes_{f_0^* \mathcal{O}_X} \mathcal{O}_{X'}$$