Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 14 (1968)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FLATNESS AND PRIVILEGE

Autor: Douady, A.

Kapitel: §1. Analytic subspaces of an analytic space

DOI: https://doi.org/10.5169/seals-42343

FLATNESS AND PRIVILEGE

by A. Douady

I. Flat Morphisms

§ 1. *Analytic subspaces of an analytic space*

Let Y_1 and Y_2 be closed analytic subspaces of an analytic space X, and let them be defined by the $\mathcal{O}_X$ ideals J_1, J_2.

Definition 1: We say that Y_1 is *analytically included* in Y_2, and we write $Y_1 \subset Y_2$, when $J_1 \supset J_2$.

Remark: The analytic inclusion implies the set theoretic inclusion, but the converse is not true.

Example: $X = (\mathbf{C}, \mathcal{O}_{\mathbf{C}})$; $J_1 = (x)$, $J_2 = (x^2)$. The space Y_1 is a simple point, Y_2 is a double point, $Y_1 \not\supset Y_2$, while they have the same underlying set.

Definition 2: The subspace $Y_1 \cup Y_2$ is the smallest subspace of X containing Y_1 and Y_2, and it is defined by $J_1 \cap J_2$. The subspace $Y_1 \cap Y_2$ is the biggest subspace of X contained in both Y_1 and Y_2, and it is defined by $J_1 + J_2$.

Remark: The underlying set of $Y_1 \cup Y_2$ (Resp. $Y_1 \cap Y_2$) is the union (Resp. intersection) of the underlying sets of Y_1 and Y_2. However $\cup$ and $\cap$ of analytic spaces do not satisfy the distributivity laws which hold in set-theory: $(Y_1 \cup Y_2) \cap Y_3$ contains $Y_1 \cap Y_3$ and $Y_2 \cap Y_3$, and therefore their union; similarly $(Y_1 \cap Y_2) \cup Y_3 \subset (Y_1 \cup Y_3) \cap (Y_2 \cup Y_3)$. In general the converse inclusions do not hold.

Example: Let $X = \mathbf{C}^2$ and Y_1, Y_2, Z be given by ideals $(x-y)$, $(x+y)$ and (x) respectively.

$(Y_1 \cup Y_2) \cap Z$ is $\{0\}$ provided with $\mathbf{C}\{y\}/(y^2)$, while $(Y_1 \cap Z) \cup (Y_2 \cap Z)$ is the reduced space $\{0\}$. On the other hand: $Y_1 \cap Y_2 \subset Z$, $(Y_1 \cap Y_2) \cup Z = Z$, while $(Y_1 \cup Z) \cap (Y_2 \cup Z)$ is the space defined by the ideal (x^2, xy). Its local ring at the origin is $\mathbf{C}\{x, y\}/(x^2, xy)$ in which x is nilpotent.

Definition 3: Let X', X be analytic spaces, Y a closed analytic subspace of X defined by J, and $f = (f_0, f^1) : X' \to X$ a morphism.

The inverse image of Y by f, $f^{-1}(Y)$, is the analytic subspace Y' of X' defined by the ideal $J' = f^1(J)\,\mathcal{O}_{X'}$.

The inverse image of a simple point x in X is called the f-fiber over x, and is denoted by $f^{-1}(x)$ or $X'(x)$.

Proposition 1: If $f = (f_0, f^1) : X' \to X$ is a morphism of analytic spaces, and Y is a subspace of X, then $f^{-1}(Y) \simeq Y \times_X X'$.

Proof: Let T be any analytic space, and $g : T \to X'$ a morphism. Then g can be considered as a morphism from T to $f^{-1}(Y)$ if and only if $f \circ g$ can be considered as a morphism from T to Y. Thus $f^{-1}(Y)$ and $X' \times_X X$ are solutions of the same universal problem.

§ 2. *Analytic pull-back*

In the following we want to generalize the notion of inverse image of a subspace.

We shall first recall the basic properties of the tensor product $E \otimes_A F$, where A is a commutative ring and E, F are two A-modules.

(1°) $E \otimes A^n = E^n$ $(n \in N)$

(2°) If the sequence of A-modules $F' \to F \to F'' \to 0$ is exact, then also the sequence $E \otimes F' \to E \otimes F \to E \otimes F'' \to 0$ is exact. (Right exactness of the tensor product)

(3°) If $(F_i)_{i \in I}$; $f_{ij} : F_j \to F_i$ is an inductive system, then

$$E \otimes \varinjlim F_i = \varinjlim (E \otimes F_i).$$

On the other hand these properties characterize completely the functor $\otimes$.

Definition 1: Let $f = (f_0\ f^1) : X' \to X$ be a morphism of analytic spaces, and $\mathcal{E}$ an $\mathcal{O}_X$-module. Then $f_0^*\mathcal{E}$ is an $f_0^*\,\mathcal{O}_X$-module and $\mathcal{O}_{X'}$ is also an $f_0^*\,\mathcal{O}_X$-module (by $f^1 : f_0^*\,\mathcal{O}_X \to \mathcal{O}_{X'}$).

The analytic pull-back $f^*\mathcal{E}$ of $\mathcal{E}$ by f is defined by scalar extension:

$$f^*\mathcal{E} = f_0^*\mathcal{E} \otimes_{f_0^*\mathcal{O}_X} \mathcal{O}_{X'}$$