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*Proof:*  $\text{Tor}_1^A(k, F) = \text{Tor}_1^B(\bar{B}, F)$ , by prop. 7.

§ 5. *Geometric applications of the flatness criterions*

A) *Flatness for finite morphisms*

*Proposition 1:* Let  $\pi: X \rightarrow S$  be a finite morphism (i.e. proper with finite fibres) of analytic spaces. Then  $\pi_*(\mathcal{O}_X)$  is a coherent analytic sheaf over  $S$ . The following conditions are equivalent:

- (a)  $\pi$  is flat (i.e. for every  $x \in X$ ,  $\mathcal{O}_{X,x}$  is a flat  $\mathcal{O}_{S,s}$ -module,  $s = \pi(x)$ ).
- (b) For every  $s$ ,  $(\pi_* \mathcal{O}_X)_s$  is a flat  $\mathcal{O}_{S,s}$ -module.
- (c)  $\pi_* \mathcal{O}_X$  is a locally free sheaf.

*Proof:* Because  $\pi$  is finite  $\pi_*(\mathcal{O}_X)_s = \bigoplus_{x \in \pi^{-1}(s)} \mathcal{O}_{X,x}$ , thus the only point to prove is (b)  $\Rightarrow$  (c).

Now if  $\mathcal{O}_{X,x}$  is a flat  $\mathcal{O}_{S,s}$ -module, then (by theorem 2)  $\mathcal{O}_{X,x}$  is free, and a coherent sheaf whose fibers are free is a locally free sheaf.

*Proposition 2:* Let  $S$  be a reduced analytic space and  $\mathcal{E}$  a coherent  $\mathcal{O}_S$ -module. Let  $E(s)$  be the finite dimensional vector space (over  $\mathbb{C}$ )  $\mathcal{E}_s \otimes_{\mathcal{O}_{S,s}} \mathbb{C}_s$ .  $\mathcal{E}$  is a locally free  $\mathcal{O}_{S,s}$ -module if and only if  $\dim_{\mathbb{C}} E(s)$  is locally constant.

*Proof:* If  $\mathcal{E}$  is locally free, then  $\dim_{\mathbb{C}} E(s)$  is locally constant. Suppose now that  $\dim_{\mathbb{C}} E(s)$  is locally constant in an open set  $U \subset S$ , and that  $\mathcal{O}_U^p \xrightarrow{d} \mathcal{O}_U^q \rightarrow \mathcal{E}_U \rightarrow 0$  is exact.  $d$  is determined by a  $p \times q$  matrix of analytic functions on  $U$ , so it gives a morphism  $\mathbb{C}_U^p \xrightarrow{d} \mathbb{C}_U^q$  of trivial vector bundles over  $U$ .

From the exact sequence  $\mathcal{O}_s^p \xrightarrow{d_s} \mathcal{O}_s^q \rightarrow \mathcal{E}_s \rightarrow 0$ , we get (by making tensor-products with  $\mathbb{C}_s$ ) the exact sequence:

$$\mathbb{C}_s^p \xrightarrow{d(s)} \mathbb{C}_s^q \rightarrow E(s) \rightarrow 0,$$

which shows that  $d$  has constant rank in  $U$ . Thus  $\text{Ker } d$  and  $\text{Im } d$  are vector bundles, and we can write

$$\mathbb{C}_U^p = F_1 \oplus G_1, \quad \mathbb{C}_U^q = F_0 \oplus G_0,$$

$$d : \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq F_0. \end{cases}$$

Now  $\mathcal{E} \simeq$  the sheaf of analytic sections of  $G_0$ , therefore  $\mathcal{E}$  is locally free.

*Definition 1:* Let  $\pi : X \rightarrow S$  be a finite morphism of analytic spaces, and  $s \in S$ . For each  $x \in X(s) = \pi^{-1}(s)$ ,  $\mathcal{O}_{X(s),x} = \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}$  is finite dimensional vectorspace over  $\mathbf{C}$ . Denote its dimension by  $v(x)$ . Then the degree  $v(s)$  of  $s$  is defined by  $v(s) = \sum_{x \in X(s)} v(x)$ .

*Theorem 1:* Let  $\pi : X \rightarrow S$  be a finite morphism of analytic space and let  $S$  be a reduced space. Then  $X$  is flat over  $S$  if and only if  $v(s)$  is locally constant function of  $s$ .

$$\begin{aligned} \text{Proof: } v(s) &= \sum_{x \in X(s)} \dim_{\mathbf{C}} \mathcal{O}_{X(s),x} = \dim_{\mathbf{C}} \left( \bigoplus_{x \in X(s)} \mathcal{O}_{X(s),x} \right) \\ &= \dim_{\mathbf{C}} \left( \bigoplus_{x \in X(s)} (\mathbf{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{X,x}) \right) \\ &= \dim_{\mathbf{C}} \mathbf{C} \otimes_{\mathcal{O}_{S,s}} \pi_* (\mathcal{O}_X)_s = \dim_{\mathbf{C}} E(s). \end{aligned}$$

The theorem follows from propositions 1 and 2.

### Examples of flat morphisms

*Example 1:* If  $\pi : X \rightarrow S$  is a local isomorphism near  $x$ , then  $\pi$  is flat at  $x$ .

*Example 2:* Consider § 2, Ex. 1. Here  $v(x) = 1$ .

### Examples of non-flat morphisms

*Examples 1:* If  $X \subset S$  is a closed subspace, not open,  $v(s)$  is not locally constant.

*Example 2:* Let  $X$  be a subspace of  $\mathbf{C}^4$  defined by the ideal intersection of  $(x_3, x_4)$  and  $(x_1 - x_1, x_4 - x_2)$  (which is equal to the product ideal) and let  $\pi$  be the projection onto the  $(x_1, x_2)$ -plane  $\mathbf{C}^2$ . Then  $X$  is a union of two 2-planes in  $\mathbf{C}^4$ , whose intersection is  $(0)$ . When  $s \neq 0$ ,  $X(s)$  consists of two simple points, so  $v(s) = 2$ .  $X(0)$  is given by the ideal  $(x_1, x_2, x_3^2, x_3x_4, x_4^2)$ , thus  $v(0) = 3$ .

*Example 3:* Let  $S = \{(u, v, w) \in \mathbf{C}^3 \mid v^2 = uw\}$  and  $\pi : \mathbf{C}^2 \rightarrow S$  be the map  $(x, y) \rightarrow (x^2, xy, y^2)$ . This map identifies  $S$  with the quotient of  $\mathbf{C}^2$  by the equivalence relation identifying  $(x, y)$  with  $(-x, -y)$ . However,  $\pi$  is not flat, since for  $s \in S$ ,  $v(s) = 2$  if  $s \neq 0$  and  $v(s) = 3$  if  $s = 0$ .

B) *Projection of a product of analytic spaces*

*Theorem 2:* Let  $S$  and  $X$  be analytic spaces. If  $\pi : S \times X \rightarrow S$  is the projection morphism, then  $\pi$  is flat, i.e.  $\mathcal{O}_{S \times X, (s,x)}$  is a flat  $\mathcal{O}_{S,s}$  module for every  $(s, x) \in S \times X$ .

To prove this theorem we need first some homological algebra. Then we shall show it in the particular case, when  $S$  is a manifold, and finally in the general case.

(a) *Koszul complex*

Let  $A$  be a ring,  $M$  an  $A$ -module and  $h_1, \dots, h_n$  homomorphisms  $M \rightarrow M$ , which commute with each other, i.e.  $h_i h_j = h_j h_i$  for every  $i, j$ .

If  $1 \leq k \leq n$ , set  $Q_k = M/h_1(M) + \dots + h_k(M)$ , and  $Q_0 = M$ , thus, in particular,  $Q_n = Q = M / \sum_{i=1}^n h_i(M)$ . Every  $h_k$  induces a map  $\tilde{h}_k : Q_{k-1} \rightarrow Q_{k-1}$ .

*Definition 2:* The sequence  $(h_1, \dots, h_n)$  is called regular if each of the mappings  $\tilde{h}_k$  ( $1 \leq k \leq n$ ) is injective.

The Koszul complex of the module  $M$  and of the mappings  $h_k$  ( $1 \leq k \leq n$ )  $K. = K. [M; h_1, \dots, h_n]$  is defined in the following way:

$$K_i = \wedge^{n+i} A^n \otimes M \simeq M^{\binom{n}{i}}, \quad 0 \leq i \leq n.$$

We define the homomorphisms  $d_i : K_i \rightarrow K_{i-1}$  ( $i > 0$ ) by  $\lambda \otimes x \rightarrow \sum_i (e_i \wedge \lambda) \otimes \otimes h_i(x)$ , where  $(e_i)$  is the natural base of  $A^n$ . We also define  $\varepsilon : K_0 \rightarrow Q$  as the natural map  $: K_0 = M \rightarrow M / \sum_{i=1}^n h_i(M) = Q$ . Using the fact that  $h_1, \dots, h_n$  commute with each other, it is easy to verify that

$$(d_{i-1} \circ d_i)(\lambda \otimes x) = \sum_{i,j} (e_j \wedge e_i \wedge \lambda) \otimes h_j(h_i(x)) = 0;$$

also  $\varepsilon d_1 = 0$ . Thus  $K.$  is really a complex.

*Theorem 3 (Poincaré-Koszul).*

If  $(h_1, \dots, h_n)$  is a regular sequence, then

$$H_i(K.) = \begin{cases} Q & \text{if } i = 0 \\ \cdot & \text{if } i = 1 \\ 0 & \text{if } i > 1 \end{cases}$$

If  $h_i \in A$ , it defines the map:  $A \xrightarrow{h_i I} A$ , which we denote also by  $h_i$ . We say that  $(h_1, \dots, h_n)$  is a regular sequence of elements if  $(h_1 I, \dots, h_n I)$  is a regular sequence.

*Corollary.* If  $(h_1, \dots, h_n)$  is a regular sequence of elements, then the Koszul complex  $K. = K. [A; h_1, \dots, h_n] = \{ \wedge^{n-1} A^n \simeq A^{(n)} \}$  is a free resolution of  $Q = A/(h_i)$  ( $(h_i)$  is the ideal generated by  $h_1, \dots, h_n$ ).

*Example:* If  $A = \mathbf{C} \{x_1, \dots, x_n\}$ ;  $h_i = x_i$ , then  $Q_k = A/(x_1, \dots, x_k) = \mathbf{C} \{x_{k+1}, \dots, x_n\}$  and  $Q = Q_n = \mathbf{C}$ . The complex  $K. = K. [A; x_1, \dots, x_n]$  is a free resolution of  $\mathbf{C}$ .

(b) *Proof of theorem 2, when  $S$  is a complex manifold*

In this case we can take  $\mathcal{O}_{S,S} = \mathbf{C} \{t_1, \dots, t_m\} = A$  and if  $\mathcal{O}_{X,x} = \mathbf{C} \{x_1, \dots, x_n\}/(f_1, \dots, f_p)$ , then

$$\mathcal{O}_{S \times X, (s,x)} = \mathbf{C} \{t_1, \dots, t_m, x_1, \dots, x_n\}/(f_1, \dots, f_p) = B.$$

$B$  is an  $A$ -module in a natural way.

By the corollary of the Poincaré-Koszul theorem  $K. = K. [A; t_1, \dots, t_m]$  in a free resolution of  $\mathbf{C}$ . We want to compute the modules  $\text{Tor}_i^A(\mathbf{C}, B) = H_i(K. \otimes B)$  ( $i > 0$ ).

It's easily seen, that we can consider the complex  $K. \otimes B$  as a Koszul

complex  $K'. = K. [B; t_1, \dots, t_m]$  (where  $t_i : B \xrightarrow{t_i I} B$ ). But now the sequence  $(t_1, \dots, t_m)$  is regular, thus by the Poincaré-Koszul theorem  $H_i[K'] = 0$  if  $i > 0$ .

In particular:  $\text{Tor}_1^A(\mathbf{C}, B) = H_1[K. \otimes B] = H_1[K'] = 0$ . By the second flatness criterion  $B$  is  $A$ -flat.

(c) *The general case*

The question being local, we can suppose that  $S \subset W \subset \mathbf{C}^n$ , where  $W$  is open, and  $S$  an analytic subspace of  $W$ . Let  $S$  be defined by  $g_1, \dots, g_r$ . Then  $S \times X \subset W \times X$  and  $\mathcal{O}_S = \mathcal{O}_W/(g_1, \dots, g_r)$ . On the other hand  $\mathcal{O}_{S \times X} = \mathcal{O}_{W \times X}/(g_1, \dots, g_r) = \mathcal{O}_S \otimes_{\mathcal{O}_W} \mathcal{O}_{W \times X}$ . The last equality follows from

the fact, that if  $\pi : X \rightarrow S$  is a morphism, and  $S' \subset S$  a subspace,  $X' = \pi^{-1}(S')$ ,

$$\text{then } \mathcal{O}_{X'} = \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathcal{O}_X.$$

*Remark* : This a particular case of the following proposition: if  $\pi$  and  $\pi'$  are two morphisms of which at least one is finite, then

$$\begin{array}{ccc} X & & Y \\ \pi \searrow & & \swarrow \pi' \\ & S & \end{array} \quad \mathcal{O}_{X \times_S Y} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_Y.$$

We have proved that  $\mathcal{O}_{W \times X}$  is  $\mathcal{O}_W$ -flat, so by scalar extension  $\mathcal{O}_{S \times X}$  is  $\mathcal{O}_S$  flat.

*Corollary* : If  $X$  and  $S$  are two manifolds and  $\pi : X \rightarrow S$  is a submersion, then  $\pi$  is flat.

### III. PRIVILEGED POLYCYLINDERS

#### § 1. Banach vector bundles over an analytic space

Let  $E$  be a Banach space and  $X$  an analytic space. We denote then by  $E_X$  the trivial bundle  $X \times E$  over  $X$ .

To define bundle morphisms, we first define the sheaf  $\mathcal{H}_X(E)$  of germs of analytic morphisms from  $X$  to  $E$ . If  $U \subset \mathbb{C}^n$  is open, then the set  $\mathcal{H}(U, E)$  of analytic morphisms from  $U$  into  $E$  consists of all functions  $g : U \rightarrow E$  having at every point  $x \in U$  a converging power series expansion.

Let now  $X'$  be a local model for  $X$ , i.e.  $X'$  is the support of the quotient sheaf  $\mathcal{O}_U/J$ , where  $U \subset \mathbb{C}^n$  is open and  $J$  is a coherent sheaf of ideals of  $\mathcal{O}_U$ , then  $\mathcal{H}_{X'}(E)$  is the sheaf associated to the presheaf  $V \rightarrow \mathcal{H}(V, E)/J_V \cdot \mathcal{H}(V, E)$  ( $V \subset U$ ,  $V$ -open).

*Remark* : If  $X'$  is reduced, the sections of  $\mathcal{H}_{X'}(E)$  are just the functions from  $X'$  to  $E$  which are locally induced by analytic functions on open sets in  $U$ .

The sheaf  $\mathcal{H}_X(E)$  is constructed with help of the local models  $X'$  of  $X$ , i.e.  $\mathcal{H}_X(E)|_{X'} = \mathcal{H}_{X'}(E)$ , for every local model  $X'$ .

*Definition 1* : The set of analytic morphisms from an analytic space  $X$  into a Banach space  $E$  is the set  $\mathcal{H}(X; E)$  of sections of the sheaf  $\mathcal{H}_X(E)$ .

Let  $\mathcal{L}(E, F)$  be the Banach space of all continuous linear mappings from the Banach space  $E$  into the Banach space  $F$ .

*Definition 2* : An analytic vector bundle morphism from  $E_X$  into  $F_X$  is an analytic morphism from  $X$  into  $\mathcal{L}(E, F)$ .