Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 14 (1968)
Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FLATNESS AND PRIVILEGE
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Kapitel: §1. Banach vector bundles over an analytic space
DOI: https://doi.org/10.5169/seals-42343

Remark : This a particular case of the following proposition: if π and π' are two morphisms of which at least one is finite, then

$$\begin{tikzcd} X \arrow[dr, "\pi"'] & & Y \arrow[dl, "\pi'"] \\ & S & \end{tikzcd} \qquad \mathcal{O}_{X \underset{S}{\times} Y} = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_y .$$

We have proved that $\mathcal{O}_{W \times X}$ is $\mathcal{O}_W$-flat, so by scalar extension $\mathcal{O}_{S \times X}$ is $\mathcal{O}_S$ flat.

Corollary : If X and S are two manifolds and $\pi : X \rightarrow S$ is a submersion, then π is flat.

III. Privileged polycylinders

§ 1. *Banach vector bundles over an analytic space*

Let E be a Banach space and X an analytic space. We denote then by E_X the trivial bundle $X \times E$ over X.

To define bundle morphisms, we first define the sheaf $\mathcal{H}_X(E)$ of germs of analytic morphisms from X to E. If $U \subset \mathbf{C}^n$ is open, then the set $\mathcal{H}(U, E)$ of analytic morphisms from U into E consists of all functions $g : U \rightarrow E$ having at every point $x \in U$ a converging power series expansion.

Let now X' be a local model for X, i.e. X' is the support of the quotient sheaf $\mathcal{O}_U/J$, where $U \subset \mathbf{C}^n$ is open and J is a coherent sheaf of ideals of $\mathcal{O}_U$, then $\mathcal{H}_{X'}(E)$ is the sheaf associated to the presheaf $V \rightarrow \mathcal{H}(V, E)/J_V \,.\, \mathcal{H}(V, E)$ ($V \subset U$, V-open).

Remark : If X' is reduced, the sections of $\mathcal{H}_{X'}(E)$ are just the functions from X' to E which are locally induced by analytic functions on open sets in U.

The sheaf $\mathcal{H}_X(E)$ is constructed with help of the local models X' of X, i.e. $\mathcal{H}_X(E)|X' = \mathcal{H}_{X'}(E)$, for every local model X'.

Definition 1 : The set of *analytic morphisms* from an analytic space X into a Banach space E is the set $\mathcal{H}(X; E)$ of sections of the sheaf $\mathcal{H}_X(E)$.

Let $\mathcal{L}(E, F)$ be the Banach space of all continuous linear mappings from the Banach space E into the Banach space F.

Definition 2 : An analytic vector bundle morphism from E_X into F_X is an analytic morphism from X into $\mathcal{L}(E, F)$.

Let E be a topological space, X an analytic space, and $\pi : E \to X$ a continuous projection.

$$\begin{tikzcd} E & E|U_\iota \arrow[l, hook'] \arrow[r, "\phi_\iota"] \arrow[dr] & E_{\iota U_\iota} \arrow[d] \\ X \arrow[from=u, "\pi"'] & & U_\iota \arrow[ll, hook'] \end{tikzcd}$$

Suppose that X has an open covering $(U_\iota)_{\iota \varepsilon I}$, and that for every $\iota \in I$ there is given a trivial Banach space bundle $E_{\iota U_\iota}$ and a homeomosphism ϕ_ι, such that the following diagram is commutative:

We suppose further that for each pair $\iota, \kappa \in I$ there is given an analytic vector bundle morphism $\gamma_{\iota\kappa} : E_{\kappa U_\iota \cap U_\kappa} \to E_{\iota U_\iota \cap U_\kappa}$, with the underlying mapping $\phi_\iota \circ \phi_\kappa^{-1}$, such that:

$$\gamma_{\iota\lambda} = \gamma_{\iota\kappa}\, \gamma_{\kappa\lambda}; \quad \gamma_{\iota\iota} = I, \qquad \text{for all} \qquad \iota, \kappa, \gamma \in I\,.$$

This data gives a Banach vector bundle atlas on E and provides E with the structure of a Banach vector bundle over X (two atlases are equivalent if there exists an atlas containing both).

Remark: If X is reduced, the $\gamma_{\iota\kappa}$ are determined by their underlying map and the condition $\gamma_{\iota\lambda} = \gamma_{\iota\kappa}\, \gamma_{\kappa\lambda}$ is automatically satisfied.

Using local triviality, we can define morphisms for general Banach vector bundles.

Proposition 1: Let $\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$ be a morphism of two Banach vector bundles E and F, and $x \in X$.

If $\phi_x \in \mathscr{L}(E(x), F(x))$ is an isomorphism, then there exists an open neighbourhood $U \subset X$ of x, such that $\phi|U : E|U \to F|U$ is a vector bundle isomorphism.

Proof: First we take a trivialisation $E|V = E_{0V}$, $F|V = F_{0V}$ at $x \in V \subset X$ (V-open).

The set $\mathrm{Isom}(E_0, F_0)$ of isomorphic mappings is an open subset of $\mathscr{L}(E_0, F_0)$ and the mapping $g \to q^{-1}$ is an analytic isomorphism:

$$\mathrm{Isom}(E_0, F_0) \simeq \mathrm{Isom}(F_0, E_0)\,.$$

So we have in an open neighbourhood $U \subset X$ of x an analytic morphism $y \to \phi_y^{-1} \in \mathscr{L}(F_0, E_0)$, which defines the inverse morphism $(\phi|U)^{-1} : F|U \to$ $\to E|U$.

Definition 3 : Let E and F be two Banach spaces and f a continuous linear mapping from E into F. f is a *split mono-(epi) morphism,* if there exists a mapping $g \in \mathscr{L}(F, E)$ such that $g \circ f = I_E$. (Resp. $f \circ g = I_F$.)

Definirion 4 : Let E_1 and E_2 be two Banach vector bundles over an analytic space X, and f a vector bundle morphism from E_1 into E_2. *f is a split mono (epi) morphism,* if there exists a vector bundle morphism $g : E_2 \to E_1$ such that $g \circ f = I_{E_1}$. (Resp. $f \circ g = I_{E_2}$.)

Equivalently, f :
$$\begin{tikzcd} E_1 \arrow[rr] \arrow[dr] & & E_2 \arrow[dl] \\ & X & \end{tikzcd}$$
is a split monomorphism if an only if E_2 can be decomposed in a direct sum $E_2 = F_2 \oplus G_2$ such that

$$f: \begin{cases} E_1 \simeq F_2 \\ 0 \to G_2 \end{cases}.$$

and f is a split epimorphism if correspondingly

$$E_1 = F_1 \oplus G_1 , \quad \text{such that} \quad f: \begin{cases} F_1 \to 0 \\ G_1 \simeq E_2 \end{cases}.$$

Proposition 2 : Let
$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$$
be a bundle morphism and $x \in X$.

If $\phi_x : E(x) \to F(x)$ is a split epi (mono) morphism, then the point x has an open neighbourhood $U \subset X$, such that $\phi|U : E|U \to F|U$ is a split vector bundle epi (mono) morphism.

Proof : Suppose that ϕ_x is a split epimorphism. We take first a trivilisation $E|V = E_{0V}$, $F|V = F_{0V}$ at x, so that there exists a mapping $\sigma \in \mathscr{L}(F_0, E_0)$, $\phi_x \circ \sigma = I_{F_0}$. If we define a morphism $\psi : F_{0V} \to E_{0V}$ by $x \to \sigma \in \mathscr{L}(F_0, E_0)$, the morphism $\gamma = \phi \circ \psi : F_{0V} \to F_{0V}$ has an isomorphic fibre mapping $\gamma_x = I_{F_0}$ in x. By proposition 1 we have an isomorphic restriction $\gamma|U$, $\phi|U \circ (\psi|U \circ (\gamma|U)^{-1}) = I_{F_{0U}}$.

When ϕ_x is a split monomorphism, the proof is similar.

Definition 5 : Let B_1, B_2, B_3 be Banach spaces, and $j, k : B_1 \xrightarrow{j} B_2 \xrightarrow{k} B_3$ continuous linear mappings. This sequence forms *a complex*, if $k \circ j = 0$. This sequence is *split exact* if the space B_i can be decomposed in direct sums $B_i = C_i \oplus D_i$ such that

$$j: \begin{cases} C_1 \to 0 \\ D_1 \simeq C_2 \end{cases} \qquad k: \begin{cases} C_2 \to 0 \\ D_2 \simeq C_3 \end{cases}.$$

Definition 6: A Banach vector bundle morphism sequence

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[dr] & E_2 \arrow[r, "g"] \arrow[d] & E_3 \arrow[dl] \\ & X & \end{tikzcd}$$

is *a complex* if $g \circ f = 0$.

The sequence is *split exact*, if every E_i can be decomposed $E_i = F_i \oplus G_i$, such that:

$$f: \begin{cases} F_1 \to 0 \\ G_1 \simeq F_2 \end{cases} \qquad g: \begin{cases} F_2 \to 0 \\ G_2 \simeq F_3 \end{cases}.$$

Theorem 1: Let
$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[dr] & E_2 \arrow[r, "g"] \arrow[d] & E_3 \arrow[dl] \\ & X & \end{tikzcd}$$
be a complex of Banach vector bundles and $x_0 \in X$.

If the sequence of Banach spaces $E_1(x_0) \overset{fx_0}{\to} E_2(x_0) \overset{fx_0}{\to} E_3(x_0)$ is split exact, then there exists an open neighbourhood $U \subset X$ of x_0, such that $E_1 | U \overset{f|U}{\to} E_2 | U \overset{g|U}{\to} E_3 | U$ is a split exact sequence of Banach vector bundles.

Proof: We take a neighbourhood V of x, such that we have a complex $E_{1V} \overset{f|V}{\to} E_{2V} \overset{g|V}{\to} E_{3V}$ of trivial bundles. By assumption we have the decompositions $E_{iV}(x_0) = F_i(x_0) \oplus G_i(x_0)$ with

$$f_{x_0}: \begin{cases} F_1(x_0) \to 0 \\ G_1(x_0) \simeq F_2(x_0) \end{cases} \qquad g_{x_0}: \begin{cases} F_2(x_0) \to 0 \\ G_2(x_0) \simeq F_3(x_0) \end{cases}.$$

By proposition 2, $f|V: G_{1V} \to E_{2V}$, $g|V: G_{2V} \to E_{3V}$ are both split monomorphisms in a neighbourhood $W \subset V$ of x_0 and the images $F_2 = f(G_{1W})$, $F_3 = g(G_{2W})$ are subbundles of E_{2W} esp. E_{3W}, such that

$$E_{2W} = F_2 \oplus G_{2W}, \quad E_{3W} = F_3 \oplus G_{3W}.$$

By our construction

$$g \mid W : \begin{cases} F_2 & \to 0 \\ G_2\, W \simeq F_3 \end{cases} .$$

If $p\colon E_{2W} \to F_2$ is the projection with kernel G_{2W}, the map, $p \circ f \colon E_{1W} \to F_2$ is a split epimorphism in x_0. Again by prop. 2 we have over an open eighbourhood $U \subset W$ of x_0 a decomposition $E_{1U} = F_1 \oplus G_{1U}$ (with $F_1 = \operatorname{Ker} p \circ f$)

$$(p \circ f) \mid U : \begin{cases} F_1 & \to 0 \\ G_{1U} \xrightarrow{\sim} F_{2U} \end{cases} .$$

The image $f|U\,(F_1)$ is contained in G_{2U}. But $g|U \circ f|U = 0$ and $g|G_{2U}$ is a monomorphism hence $f|U : F_1 \to 0$. We get finally (restricting all our morphisms to U)

$$f \mid U : \begin{cases} F_{1U} \to 0 \\ G_{1U} \simeq F_{2U} \end{cases} \qquad g \mid U : \begin{cases} F_{2U} \to 0 \\ G_{2U} \xrightarrow{\sim} F_{3U} \end{cases} .$$

§ 2. *Privileged polycylinders*

Definition 1: A polycylinder in $\mathbf{C}^n$ is a compact set K of the form $K = K_1 \times \ldots \times K_n$ where each K_i is a compact, convex subset of $\mathbf{C}$, with nonempty interior. If each K_i is a disc, then K is a polydisc. We first recall the following theorem of Cartan.

Theorem 1: Let K be a polycylinder contained in an open subset U of $\mathbf{C}^n$. Let $\mathscr{F}$ be a coherent analytic sheaf on U.

(A) There exists an open neighbourhood of K over which $\mathscr{F}$ admits a finite free resolution

$$0 \to \mathscr{L}_n \to \ldots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0 .$$

(B) $H^q(K, \mathscr{F}) = 0$ for $q > 0$.

(Reference: For instance Gunning and Rossi.)

We have the following consequences of this theorem:

1) Given a finite free resolution

$$0 \to \mathscr{L}_n \to \ldots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0$$

of a coherent sheaf $\mathscr{F}$, the sequence