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$$g \mid W : \begin{cases} F_2 & \to 0 \\ G_2 W \simeq F_3 \end{cases} .$$

If p: $E_{2W} \to F_2$ is the projection with kernel G_{2W}, the map, $p \circ f$: $E_{1W} \to F_2$ is a split epimorphism in x_0. Again by prop. 2 we have over an open eighbourhood $U \subset W$ of x_0 a decomposition $E_{1U} = F_1 \oplus G_{1U}$ (with $F_1 = \operatorname{Ker} p \circ f$)

$$(p \circ f) \mid U : \begin{cases} F_1 & \to 0 \\ G_{1U} & \overset{\sim}{\to} F_{2U} \end{cases} .$$

The image $f|U(F_1)$ is contained in G_{2U}. But $g|U \circ f|U = 0$ and $g|G_{2U}$ is a monomorphism hence $f|U : F_1 \to 0$. We get finally (restricting all our morphisms to U)

$$f \mid U : \begin{cases} F_{1U} \to 0 \\ G_{1U} \simeq F_{2U} \end{cases} \qquad g \mid U : \begin{cases} F_{2U} \to 0 \\ G_{2U} \overset{\sim}{\to} F_{3U} \end{cases} .$$

§ 2. *Privileged polycylinders*

Definition 1: A polycylinder in $\mathbf{C}^n$ is a compact set K of the form $K = K_1 \times \ldots \times K_n$ where each K_i is a compact, convex subset of $\mathbf{C}$, with nonempty interior. If each K_i is a disc, then K is a polydisc. We first recall the following theorem of Cartan.

Theorem 1: Let K be a polycylinder contained in an open subset U of $\mathbf{C}^n$. Let $\mathscr{F}$ be a coherent analytic sheaf on U.

(A) There exists an open neighbourhood of K over which $\mathscr{F}$ admits a finite free resolution

$$0 \to \mathscr{L}_n \to \ldots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0 .$$

(B) $H^q(K, \mathscr{F}) = 0$ for $q > 0$.
(Reference: For instance Gunning and Rossi.)
We have the following consequences of this theorem:

1) Given a finite free resolution

$$0 \to \mathscr{L}_n \to \ldots \to \mathscr{L}_1 \to \mathscr{L}_0 \to \mathscr{F} \to 0$$

of a coherent sheaf $\mathscr{F}$, the sequence

$$0 \to \mathcal{L}_n(K) \to \ldots \to \mathcal{L}_0(K) \to \mathcal{F}(K) \to 0$$

is an $\mathcal{O}_U(K)$ - free resolution of $\mathcal{F}(K)$.

2) Given a short exact sequence of coherent sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

then the sequence

$$0 \to \mathcal{F}_{\prime}(K) \to \mathcal{F}(K) \to \mathcal{F}''(K) \to 0 \quad \text{is exact.}$$

Let $\mathcal{F}$ be a coherent analytic sheaf on U, and let $K \subset U$ be a polycylinder If V is an open neighbourhood of K, then $\mathcal{F}(V)$ can be equipped with a Fréchet-space structure (see: Malgrange).

Hence we can give $\mathcal{F}(K)$ the structure of inductive limit of Fréchet-spaces. It is however essential for certain purposes to have Banach-spaces. This can be obtained by choosing a space slightly different from $\mathcal{F}(K)$ and by choosing K in a " privileged " way.

Let $B(K) = \{f : K \to \mathbf{C} | f$ continuous on K and analytic on $\mathring{K}\}$, then $B(K)$ is Banach algebra and $B(K) \subset C(K)$. The sections of $\mathcal{O}_U$ over K are elements of $B(K)$, and $B(K)$ is in fact the uniform closure of $\mathcal{O}_U(K)$ in $C(K)$.

If $\mathcal{L} = \mathcal{O}_U^r$, we define $B(K, \mathcal{L}) = B(K)^r$. Then $B(K; \mathcal{L})$ is a free $B(K)$-module, and since $\mathcal{L}(K) = \mathcal{O}_U(K)^r$, we have $B(K; \mathcal{L}) = B(K) \otimes_{\mathcal{O}_{U(K)}} \mathcal{L}(K)$.

We now assume that $\mathcal{F}$ is a coherent sheaf on U, where $U \subset \mathbf{C}^n$ is open. Consider a free resolution

$$(R) \qquad 0 \to \mathcal{L}_n \to \ldots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0 \quad \text{of } \mathcal{F}.$$

From (R) we get an $\mathcal{O}_U(K)$-free resolution of $\mathcal{F}(K)$

$$(R') \qquad 0 \to \mathcal{L}_n(K) \to \ldots \to \to_1(K) \to \mathcal{L}_0(K) \to \mathcal{F}(K) \to 0.$$

Taking the tensorproduct $B(K) \otimes_{\mathcal{O}_U(K)}$ we get the complex

$$B(K; \mathcal{L}.) : 0 \to B(K; \mathcal{L}_n) \to \ldots \to B(K; \mathcal{L}_1) \to B(K; \mathcal{L}_0).$$

Definition 2: The polycylinder K is called *$\mathcal{F}$-privileged* if the complex $B(K; \mathcal{L}.)$ is split-exact in every degree >0.

Remark: The property of being $\mathcal{F}$-privileged is independent of the resolution (R).

The exactnes of $B(K; \mathcal{L})$ can be expressed by $\mathrm{Tor}_i^{\mathcal{O}(K)}(B(K), \mathcal{F}(K)) = 0$, for every $i > 0$, and Tor is independent of the resolution (R). It is a little more complicated to show, that the splitting property is independent of (R), and this is omitted.

Since $B(K; \mathscr{L}_i)$ is a Banach space, the image and its complement are thus Banach spaces if K is $\mathscr{F}$-privileged. In this case we define $B(K; \mathscr{F}) = \operatorname{Coker}(B(K, \mathscr{L}_1) \to B(K; \mathscr{L}_0)) = B(K) \otimes_{\mathcal{O}_U} \mathscr{F}(K)$ and we get a $B(K)$-module, which is a Banach-space.

Warning: In the definition of split-exactnes, the subspaces are splitting vector spaces, but they are not splitting $B(K)$-modules in general.

We have the following important theorem about the existence of privileged polycylinders:

Theorem 2: Let U be an open subset of $\mathbf{C}^n$, and let $\mathscr{F}$ be a coherent analytic sheaf on U. For any $x \in U$ there exists a fundamental system of neighbourhoods of x in U, which are $\mathscr{F}$-privileged polycylinders.

For the proof, see Douady: § 7, 4, th 1.

Example: (Curves in $\mathbf{C}^2$) Let $U \subset \mathbf{C}^2$ be an open connected neighbourhood of the origin, and let $h \colon U \to \mathbf{C}$ be analytic and $h \neq 0$.

Let X be the curve given by h, that is $X = h^{-1}(0)$, $\mathcal{O}_X = \mathcal{O}_U/(h)$. We have an exact sequence $0 \to \mathcal{O}_U \xrightarrow{h.I} \mathcal{O}_U \to \mathcal{O}_X \to 0$. Consider a polycylinder $K = K_1 \times K_2 \subset U$. By definition K is $\mathcal{O}_X$-priviledged if and only if $h \colon B(K) \to B(K)$ is a split monomorphism.

Let $\dot{K}_j$ denote the boundary of K_j, and define $\ddot{K} = \dot{K}_1 \times \dot{K}_2$ ($\ddot{K}$ is called the Šilov Boundary of K).

Proposition 1: (a) The following conditions are equivalent:

(i) $h \colon B(K) \to B(K)$ is a monomorphism.

(i′) $\exists\, a > 0$ such that $\|hf\| \geqq a\|f\|$, $\forall f \in B(K)$.

(ii) $X \cap \ddot{K} = \varnothing$.

(b) If $(K_1 \times \dot{K}_2) \cap X = \varnothing$, then h is a split monomorphism (i.e. K is $\mathcal{O}_X$ privileged).

Proof: (a) (i) $\Leftrightarrow$ (i′) is a well known fact from the theory of normed vector spaces.

(ii) $\Rightarrow$ (i′). Assume $X \cap \ddot{K} = \varnothing$. If $f \in B(K)$, then it follows from the maximum principle that $\|f\| = \sup\limits_K |f(x)| = \sup\limits_{\ddot{K}} |f(x)|$. Since $h(x) \neq 0$

whenever $x \in \ddot{K}$, we get $a = \inf_K |h(x)| > 0$. Hence $\|hf\| = \sup_K |hf(x)| \geqq$ $\geqq a \sup_K |f(x)| = a \|f\|$.

(i') $\Rightarrow$ (ii). Suppose that $X \cap \ddot{K} \neq \varnothing$ and $x = (x_1, x_2) \in X \cap \ddot{K}$. We choose an analytic function $f_1 : U_1 \to \mathbf{C}$, where $U_1 \supset K_1$, and U_1 is open, such that $f_1(x_1) = 1$, $|f_1(z)| < 1$ if $z \in K_1$, $z \neq x_1$. Similarly we choose an analytic function $f_2 : U_2 \to \mathbf{C}$, with the same properties. Consider the function $f \in B(K) : (z_1, z_2) \to f_1(z_1) f_2(z_2)$. Since $h(x) = 0$ it follows that the sequence $\{h f^n\}$ converges pointwise to 0 in K.

Applying Dini's theorem we get $\|h f^n\| \to 0$. From the inequality $a \|f^n\| \leqq$ $\leqq \|h f^n\|$ we get $\|f^n\| \to 0$, which is a contradiction, because for every $n : f^n(x) = 1$.

(b) Use the Weierstrass preparation theorem (extended form).

Question. Does the condition (ii) imply that $h : B(K) \to B(K)$ is a split monomorphism?

IV. Flatness and privilege

§ 1. *Morphisms from an analytic space into $B(K)$*

Let S be an analytic space and K a polycylinder in an open set $U \subset \mathbf{C}^n$. We want to construct an $\mathcal{O}_S$-algebra homomorphism $\phi : \mathcal{O}_{S \times U}(S \times U) \to$ $\to \mathcal{H}(S; B(K))$.

(*a*) Consider first $S = U' \subset \mathbf{C}^m$, U'-open. If $h \in \mathcal{O}_{U' \times U}(U' \times U)$ and $s \in U'$, $x \in K$, define $(\phi(h)(s))(x) = h(s,x)$. Using the Cauchy integral, one can show that $\phi(h)$ is analytic. On the other hand its obvious that ϕ is an $\mathcal{O}_{U'}$-algebra homomorphism.

(*b*) Let S have a special model in the polydisc Δ in $\mathbf{C}^m$, defined by a sheaf $\mathcal{J}$ of ideals of $\mathcal{O}_\Delta$, and let $\mathcal{J}$ be generated by $f_1, \ldots, f_p$, V-a polycylinder neighbourhood of K in U. By Cartan's theorem B for a polycylinder,

the sequence $0 \to \mathcal{J}(\Delta \times V) \to \mathcal{O}(\Delta \times V) \xrightarrow{\pi} \mathcal{O}(S \times V) \to 0$ is exact. If we denote by $\tilde{\pi}$ the projection $\mathcal{H}(\Delta, B(K)) \to \mathcal{H}(S, B(K))$, $(f_1, \ldots, f_p) \cdot \mathcal{H}(\Delta, B(K)) \subset$ $\subset \operatorname{Ker} \tilde{\pi}$. Therefore, because π is surjection, there exists a unique

$\phi : \mathcal{O}(S \times V) \to \mathcal{H}(S, B(K))$, such that the diagram