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$$\mathcal{O}(\Delta \times V) \xrightarrow{\phi} \mathcal{H}(\Delta, B(K))$$

$$\pi \downarrow \qquad \qquad \downarrow \tilde{\pi}$$

$$\mathcal{O}(S \times V) \xrightarrow{\phi} \mathcal{H}(S, B(K))$$

is commutative; ϕ is evidently an \mathcal{O}_S -algebra homomorphism.

§ 2. *The flatness and privilege theorem*

Notation

Let S be an analytic space, U an open set in \mathbb{C}^n , and $\pi : S \times U \rightarrow S$ the first projection.

If \mathcal{F} is an $\mathcal{O}_{S \times U}$ module, then for every $s \in S$ we denote by $\mathcal{F}(s)$ the \mathcal{O}_U -module $i_s^* \mathcal{F}$, where i_s is the injective morphism $x \rightarrow (s, x)$ from U into $S \times U$. If $x \in U$

$$(\mathcal{F}(s))_x \simeq \mathcal{F}_{(s, x)} / m_s \cdot \mathcal{F}_{(s, x)} \simeq \mathcal{F}_{(s, x)} \otimes_{\mathcal{O}_{S, s}} \mathbb{C}_s.$$

Theorem 1: Let \mathcal{E} be a coherent and S -flat $\mathcal{O}_{S \times U}$ -module, and K a poly-cylinder in U .

(a) When K is privileged for $\mathcal{E}(s_0)$, s_0 has a neighbourhood V such that K is $\mathcal{E}(s)$ -privileged for each $s \in V$. In other words: the set $S' = \{s \in S \mid K \text{ is } \mathcal{E}(s)\text{-privileged}\}$ is open in S .

(b) It is possible to define a Banach vector bundle over S' whose fibre at any $s \in S'$ is $B(K, \mathcal{E}(s))$.

To prove the theorem we need:

Lemma 1: Under the conditions of the theorem, we can, for every $s \in S$, find a neighbourhood W of $\{s\} \times K$ and a free resolution of finite length

$$0 \rightarrow \mathcal{L}_p \xrightarrow{d_p} \dots \xrightarrow{d_2} \mathcal{L}_1 \xrightarrow{d_1} \mathcal{L}_0 \xrightarrow{\varepsilon} \mathcal{E} \rightarrow 0 \text{ in } W.$$

Proof: Let (s, x) be a point of $S \times U$ and \mathcal{L}_*^0 a finite resolution of $\mathcal{F}(x)$ in a neighbourhood of x (there exists such one, by the theorem of syzygies). We shall show that there exists a resolution \mathcal{L}^* of \mathcal{F} in a neighbourhood of (s, x) such that $\mathcal{L}^*(s) = \mathcal{L}_*^0$; if $\mathcal{L}_i^0 = \mathcal{O}_x^{r_i}$ define

$$\mathcal{L}_i = \mathcal{O}_{S \times U}^{r_i} \text{ and } \mathcal{K}_i^0 = \text{Ker } d_i^0 : \mathcal{L}_i^0 \rightarrow \mathcal{L}_{i-1}^0.$$

We shall construct by induction (with respect to i) $d_i : \mathcal{L}_1 \rightarrow \mathcal{L}_{i-1}$ in a neighbourhood of (s, x) such that $d_i(s) = d_i^0$, and prove that $\mathcal{K}_i = \text{Ker } d_i$ is S -flat and that $\mathcal{K}_i(s) = \mathcal{K}_i^0$.

$$\begin{array}{ccc} \mathcal{L}_{i+1} & \xrightarrow{d_{i+1}} & \mathcal{K}_i \\ \downarrow & & \downarrow \\ \mathcal{L}_{i+1}^0 & \xrightarrow{d_{i+1}^0} & \mathcal{K}_i^0 \end{array}$$
 Suppose that we have constructed d_i and proved the properties for \mathcal{K}_i . We can construct $d_{i+1} : \mathcal{L}_{i+1} \rightarrow \mathcal{L}_i$ in a neighbourhood of (s, x) such that the diagram is commutative.

Nakayama's lemma shows that $\text{Im } d_{i+1} = \mathcal{K}_i$ at the point (s, x) , therefore in a neighbourhood of that point.

The exact sequence

$$0 \rightarrow \mathcal{K}_{i+1} \rightarrow \mathcal{L}_{i+1} \rightarrow \mathcal{K}_i \rightarrow 0,$$

where \mathcal{K}_i and \mathcal{L}_{i+1} are S -flat, shows that \mathcal{K}_{i+1} is S -flat, and that $\mathcal{K}_{i+1}(s) = \mathcal{K}_{i+1}^0$. The first step of the induction is analogous.

Proof of the theorem: Let $s_0 \in S$ and

$$0 \rightarrow \mathcal{L}_p \xrightarrow{d_p} \dots \xrightarrow{d_1} \mathcal{L}_0 \rightarrow \mathcal{E}|_W \rightarrow 0$$

be a free $\mathcal{O}_{S \times U}$ resolution of \mathcal{E} in a neighbourhood $W = V_1 \times V_2$ of $\{s_0\} \times K$. The sheaf \mathcal{E} is \mathcal{O}_S -flat, so for each $s \in V_1$, the sequence

$$0 \rightarrow \mathcal{L}_p \otimes_{\mathcal{O}_{S|V_1}} \mathbf{C}_s \rightarrow \dots \rightarrow \mathcal{L}_1 \otimes_{\mathcal{O}_{S|V_1}} \mathbf{C}_s \rightarrow \mathcal{L}_0 \otimes_{\mathcal{O}_{S|V_1}} \mathbf{C}_s \rightarrow \mathcal{E}|_W \otimes_{\mathcal{O}_{S|V_1}} \mathbf{C}_s \rightarrow 0$$

is exact. So the sequence

$$(A) \quad 0 \rightarrow \mathcal{L}_p(s) \xrightarrow{d_p(s)} \dots \xrightarrow{d_1(s)} \mathcal{L}_0(s) \rightarrow \mathcal{E}(s)|_{V_2} \rightarrow 0$$

is exact when $s \in V_1$. Now $\mathcal{L}_i(s) \simeq \mathcal{O}_{V_2}^{r_i}$ ($0 \leq i \leq p$) and every $d_i(s)$ induces a continuous linear map:

$B(K, \mathcal{L}_i(s)) \rightarrow B(K, \mathcal{L}_{i-1}(s))$, which we also denote by $d_i(s)$. We can consider $d_i = (d_{ijk})$ as an $r_i \times r_{i-1}$ -matrix with entries from $\mathcal{O}_{S \times U}(W)$.

By § 1 we have a \mathcal{O}_S -algebra homomorphism

$$\mathcal{O}_{S \times W}(S \times W) \rightarrow \mathcal{H}(S, B(K)).$$

From the matrix (d_{ijk}) we get by this homomorphism a morphism \tilde{d}_i :

$$V_0 \rightarrow \mathcal{L}(B(K)^{r_i}, B(K)^{r_{i-1}}) = \mathcal{L}(B(K, \mathcal{L}_i(s)), B(K, \mathcal{L}_{i-1}(s))).$$

(Here V_0 is some neighbourhood of s_0) such that $\tilde{d}_i(s) = d_i(s)$ for each $s \in V_0$. In other words we have a sequence of Banach vector bundle morphisms

$$(B) \quad 0 \rightarrow B(K, \mathcal{L}_p) \xrightarrow{d_p} \dots \xrightarrow{\tilde{d}_1} B(K, \mathcal{L}_0).$$

Using the fact that $\mathcal{O}_{S \times U}(S \times U) \rightarrow \mathcal{H}(S, B(K))$ is an \mathcal{O}_S -algebra homomorphism, it easily follows that (B) is complex of Banach vector bundles over S .

Now K is $\mathcal{E}(s_0)$ -privileged, thus

$$0 \rightarrow B(K, \mathcal{L}_p(s_0)) \xrightarrow{d_p(s_0)} \dots \xrightarrow{d_1(s_0)} B(K, \mathcal{L}_0(s_0))$$

is split exact, so by theorem III.1

$$0 \rightarrow B(K, \mathcal{L}_p)|_V \xrightarrow{\tilde{d}_p|_V} \dots \xrightarrow{\tilde{d}_1|_V} B(K, \mathcal{L}_0)|_V$$

is split exact for some neighbourhood V of s_0 .

Because $\tilde{d}_i(s) = d_i(s)$ and the sequence (A) is exact part (a) of the theorem follows.

(b) $B(K, \mathcal{L}_0)|_V$ splits as the direct sum of $\text{im } \tilde{d}_1$ and a bundle E_V , such that $E_{V,s} \simeq B(K, \mathcal{E}(s))$, for each $s \in V$. We must show that these bundle structures fit together globally.

Suppose therefore that V is open in S' and that

$$\begin{aligned} 0 \rightarrow \mathcal{L}_p \xrightarrow{d_p} \dots \xrightarrow{d_2} \mathcal{L}_1 \xrightarrow{d_1} \mathcal{L}_0 \rightarrow \mathcal{E}|_{V \times V_2} \rightarrow 0 \\ 0 \rightarrow \mathcal{L}'_p \xrightarrow{d'_p} \dots \xrightarrow{d'_2} \mathcal{L}'_1 \xrightarrow{d'_1} \mathcal{L}'_0 \rightarrow \mathcal{E}|_{V \times V_2} \rightarrow 0 \end{aligned}$$

are free resolutions of ξ over $V \times V_2$.

If V_1, V_2 are open polycylinders, we can find an $\mathcal{O}_{S \times U}$ -homomorphism $\phi_0 : \mathcal{L}'_0 \rightarrow \mathcal{L}_0$ such that

$$\begin{array}{ccc} \mathcal{L}'_0 & \xrightarrow{\varepsilon'} & \mathcal{E}|_{V \times V_2} \rightarrow 0 \\ \phi_0 \uparrow & \parallel & \\ \mathcal{L}_0 & \xrightarrow{\varepsilon} & \mathcal{E}|_{V \times V_2} \rightarrow 0 \end{array}$$

commutes. ϕ_0 determines a bundle morphism $\tilde{\phi}_0: B(K, \mathcal{L}_0) \rightarrow B(K, \mathcal{L}'_0)$. $B(K, \mathcal{L}_0)$ (resp. $B(K, \mathcal{L}'_0)$) splits as $(\text{im } \tilde{d}_1) \otimes E_V$ [Resp. $(\text{im } \tilde{d}'_1) \otimes E'_V$].

Let p' be the projection morphism: $B(K, \mathcal{L}'_0) \rightarrow E'_V$ with kernel $\text{im } \tilde{d}'_1$, and put $\tilde{\phi} = p' \circ \phi_0|_{E_V}$.

The commutative diagram

$$\begin{array}{ccc}
 B(K, \mathcal{L}_0(s)) & \xrightarrow{\tilde{\phi}_0} & B(K, \mathcal{L}'_0(s)) \\
 \varepsilon \downarrow & \swarrow & \downarrow \varepsilon' \\
 & E_{V,s} & \xrightarrow{\tilde{\phi}} E'_{V,s} \\
 & \swarrow \varepsilon \simeq \alpha \circ \varepsilon & \searrow \varepsilon' \simeq \alpha' \circ \varepsilon' \\
 B(K, \mathcal{E}(s)) & \xleftarrow{\text{id}} & B(K, \mathcal{E}'(s))
 \end{array}$$

and the open mapping theorem shows that $\tilde{\phi}(s)$ is an isomorphism of Banach spaces for each $s \in V$, so $\tilde{\phi}: E_V \rightarrow E'_V$ is a bundle isomorphism. We also notice that $\tilde{\phi}$ depends only on the choice of splittings in $B(K, \mathcal{L}_0)$ and $B(K, \mathcal{L}'_0)$, and not on the choice of $\tilde{\phi}_0$. This ends the proof of the theorem.

Remark: Consider the general situation where X and S are analytic spaces, and $\pi: X \rightarrow S$ is a morphism, \mathcal{E} an \mathcal{O}_X -module. To study the local dependence of \mathcal{E} on S , one can imbed an open set X' in X in the open set $U \subset \mathbb{C}^n$. The morphism $\phi: X' \rightarrow U, \pi: X' \rightarrow S$ determine the imbedding $\pi \times \phi: X' \rightarrow S \times U$ such that the diagram commutes. \mathcal{E} can be extended by zero into a sheaf \mathcal{E}' over $U \times S$. Obviously this sheaf \mathcal{E}' is S -flat iff \mathcal{E} is S -flat.

Therefore theorem 1 makes clear also this general situation.

Corollary: If $\pi: X \rightarrow S$ is a morphism and \mathcal{E} a coherent \mathcal{O}_X -module. Then $\pi|_{\text{Supp}(\mathcal{E})}$ is an open map.

Proof: Suppose as above that X is imbedded in $S \times U$, and \mathcal{E} is extended by zero to $S \times U$. Let $x_0 \in \text{Supp } \mathcal{E}$, and V be a neighbourhood of x_0 in $S \times U$. Let $s_0 = \pi(x_0)$ and choose an $\mathcal{E}(s_0)$ -privileged polycylinder K in U , such that $\{s_0\} \times K \subset V$, over some neighbourhood W of s_0 . We have the Banach bundle $B(K, \mathcal{E}|_{\pi^{-1}(W)})$, whose fiber over s is $B(K, \mathcal{E}(s))$. Since $x_0 \in \text{Supp } \mathcal{E}(s_0)$ and K is a neighbourhood of x_0 , $B(K; \mathcal{E}(s_0)) \neq 0$. As all the fibers are isomorphic, then for all $s \in U$, $B(K; \mathcal{E}(s)) \neq 0$ and therefore $\{s\} \times K \cap \text{Supp } \mathcal{E} \neq \emptyset$, and $s \in \pi(\text{Supp } \mathcal{E})$. This proves that π is open.