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image sheaf of S of dimension l. Our main problem is to decide whether $\psi_{(l)}(S)$ is a coherent analytic sheaf of $\mathcal{O}_Y$-modules if S is a coherent analytic sheaf on X.

A VERY SPECIAL CASE

We shall consider a special case where our main problem is easily solved. Let X_0 be a compact analytic manifold of pure dimension $m - n$. We put $E^n(\rho_0) = \{ (t_1 \dots t_n) \in C^n \; ; \; |t_i| < \rho_i^0 \}$. Here $\rho_0 = (\rho_1^0 \dots \rho_n^0)$ is a fixed n-tuple of strictly positive numbers. Let $X = E^n(\rho_0) \times X_0$ and $X(\rho) = E^n(\rho) \times X_0$ for $\rho \leqslant \rho_0$. We see that X is an analytic manifold of pure dimension m. Let $\psi : X \to E^n(\rho_0)$ be the projection map. Now X is fibered by the fibers $\psi^{-1}(t) = X(t) = \{ t \} \times X_0 \cong X_0$ for $t \in E^n(\rho_0)$. We take the sheaf S to be $S = (q\mathcal{O})_X$. With these notations we can state the following.

Theorem: The direct image sheaf $\psi_{(l)}((q\mathcal{O})_X)$ is a coherent sheaf of $\mathcal{O}_{E^n(\rho_0)}$-modules for every $l \geqslant 0$.

Proof. Because X_0 is a compact analytic manifold we can find a finite Stein covering $\mathfrak{U} = \{ U_1 \dots U_{\iota*} \}$ of X_0. Let us put $\hat{U}_\iota = E^n(\rho_0) \times U_\iota$, then we see that $\hat{\mathfrak{U}} = \{ \hat{U}_1 \dots \hat{U}_{\iota*} \}$ is a Stein covering of X. Let $\hat{\xi} = \{ \hat{\xi}_{\iota_0 \dots \iota_l} \} \in C^l(\hat{\mathfrak{U}}, (q\mathcal{O})_X)$. Now $\hat{\xi}_{\iota_0 \dots \iota_l}$ is a q-tuple of holomorphic functions on $E^n(\rho_0) \times U_{\iota_0 \dots \iota_l}$. Hence $\hat{\xi}_{\iota_0 \dots \iota_l}$ admits a Taylor series of the form $\hat{\xi}_{\iota_0 \dots \iota_l} = \sum_{|\nu|=0}^{\infty} \xi^{(\nu)}_{\iota_0 \dots \iota_l} (t/\rho_0)^\nu$ where $\nu = (\nu_1, \dots, \nu_n)$, $|\nu| = \nu_1 + \dots + \nu_n$ and $(t/\rho)^\nu = (t_1/\rho_1)^{\nu_1} \dots (t_n/\rho_n)^{\nu_n}$. The uniqueness of a Taylor series shows that $\{ \xi^{(\nu)}_{\iota_0 \dots \iota_l} \}$ is an alternating cochain over $\mathfrak{U}$. Putting $\xi_{(\nu)} = \{ \xi^{(\nu)}_{\iota_0 \dots \iota_l} \} \in C^l(\mathfrak{U}, (q\mathcal{O})_X)$ we may write $\hat{\xi} = \sum \xi_{(\nu)} (t/\rho)^\nu$. Introducing the map $(\nu) : \hat{\xi} \to \xi_{(\nu)}$ we get a commutative diagram of the form:

$$\begin{array}{ccc} C^l(\hat{\mathfrak{U}}, (q\mathcal{O})_X) & \xrightarrow{\delta} & C^{l+1}(\hat{\mathfrak{U}}, (q\mathcal{O})_X) \\ (\nu)\downarrow & & \downarrow(\nu) \\ C^l(\mathfrak{U}, (q\mathcal{O})_{X_0}) & \xrightarrow{\delta} & C^{l+1}(\mathfrak{U}, (q\mathcal{O})_{X_0}) . \end{array}$$

We now need a *theorem of Cartan-Serre:* Let X_0 be a compact analytic manifold. Then, for any coherent analytic sheaf S the set $H^p(X_0, S)$ is a finite dimensional vector space for all $p \geqslant 0$.

Using this theorem we conclude that $H^l(X_0, (q\mathcal{O})_{X_0})$ has a finite base $\mathfrak{b}_1 \dots \mathfrak{b}_r$. By Leray's theorem we also have $H^l(\mathfrak{U}, (q\mathcal{O})_{X_0}) \cong H^l(X_0, (q\mathcal{O})_{X_0})$. Hence we can find $\mathfrak{b}_1 \dots \mathfrak{b}_r \in Z^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$ such that $\mathfrak{b}_\nu$ maps into $\mathbf{b}_\nu$ under the natural homomorphism $Z^l(\mathfrak{U}, (q\mathcal{O})_{X_0}) \to H^l(X_0, (q\mathcal{O})_{X_0})$. We now introduce a pseudonorm in $C^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$ as follows:

Norm definition. Let $\eta \in C^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$. Then we put $\|\eta\| = \sup_{(\iota_0 \cdots \iota_l)} \|\eta_{\iota_0 \cdots \iota_l}\|$ and $\|\eta_{\iota_0 \cdots \iota_l}\| = \max_{1 \leq \varrho \leq q} \sup |\eta_\varrho(U_{\iota_0 \cdots \iota_l})|$, where, $\eta_{\iota_0 \cdots \iota_l} = (\eta_1, \dots, \eta_q)$. Notice that it may happen that $\|\eta\| = +\infty$. Let $\mathfrak{V} = \{V_1 \dots V_{\iota^*}\}$ be an open covering of X_0. The covering $\mathfrak{V}$ is much finer than $\mathfrak{U} = \{U_1 \dots U_{\iota^*}\}$ if $V_\iota \subset\subset U_\iota$ holds for every ι. We write $\mathfrak{V} \ll \mathfrak{U}$ in that case. Let us now choose Stein coverings $\mathfrak{V}_1$ and $\mathfrak{V}$ such that $\mathfrak{V}_1 \ll \mathfrak{V} \ll \mathfrak{U}$. In $C^l(\mathfrak{V}, (q\mathcal{O})_{X_0})$ and $C^l(\mathfrak{V}_1, (q\mathcal{O})_{X_0})$ we introduce a pseudonorm just as in $C^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$. If $\xi \in C^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$ we have defined $\xi \mid \mathfrak{V} \in C^l(\mathfrak{V}, (q\mathcal{O})_{X_0})$. It follows that $\|\xi \mid \mathfrak{V}\| < \infty$ because $V_{\iota_0 \cdots \iota_l} \subset\subset U_{\iota_0 \cdots \iota_l}$. Let us now choose $\xi \in Z^l(\mathfrak{V}, (q\mathcal{O})_{X_0})$. Since $\mathfrak{b}_1 \dots \mathfrak{b}_r \in Z^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$ constitute a base of $H^l(X_0, (q\mathcal{O})_{X_0})$ it follows from Leray's theorem that $\xi = \sum a_\nu \mathfrak{b}_\nu | \mathfrak{V} + \delta\eta$ where $a_\nu \in C^1$ and $\eta \in C^{l-1}(\mathfrak{V}, (q\mathcal{O})_{X_0})$. Now we need the following.

Lemma: There exists a constant K such that $|a_\nu| \leqslant K\|\xi\|$ and $\|\eta \mid \mathfrak{V}_1\| \leqslant K\|\xi\|$.

The proof follows because by the Banach theorem the map $(a_1, \dots, a_r, \eta) \to \xi$ of the Fréchet spaces $C^r \times C^{l-1}(\mathfrak{V}, q\mathcal{O}_{X_0})$ onto $Z^l(\mathfrak{V}, (q)\mathcal{O}_{X_0}))$ is open.

Let $\xi \in C^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$. We can extend each $\xi_{\iota_0 \cdots \iota_l} \in q\,I(U_{\iota_0 \cdots \iota_l})$ constantly over $\hat{U}_{\iota_0 \cdots \iota_l} = E^n(\rho_0) \times U_{\iota_0 \cdots \iota_l}$. We get $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}, (q\mathcal{O})_X)$ obtained from ξ by a constant extension. In particular we extend $\mathfrak{b}_1 \dots \mathfrak{b}_r$ constantly to $\hat{\mathfrak{b}}_1 \dots \hat{\mathfrak{b}}_r \in Z^l(\hat{\mathfrak{U}}, (q\mathcal{O})_X)$. Let $\hat{\mathbf{b}}_1 \dots \hat{\mathbf{b}}_r$ be the images of $\hat{\mathfrak{b}}_1 \dots \hat{\mathfrak{b}}_r$ in the direct image sheaf $\psi_{(l)}((q\mathcal{O})_X)$. Let now $\xi_0 \in \psi_{(l)}((q\mathcal{O})_X)_{(0)}$ where 0 is the origin of $E^n(\rho_0)$. By definition we can find $\xi \in H^l(X(\rho_1), q\mathcal{O})$ with $0 < \rho_1 \leqslant \rho_0$ which maps into ξ_0. Now $\hat{\mathfrak{U}}(\rho_1) = \{E^n(\rho_1) \times U_\iota\}$ is a Stein covering of $X(\rho_1)$. Hence Leray's theorem shows that we can find $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho_1), (q\mathcal{O})_X)$ such that $\hat{\xi}$ maps into ξ_0. Let us write $\hat{\xi} = \sum \xi_{(\nu)}(t/\rho_1)^\nu$ where $\xi_{(\nu)} \in Z^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$. Let us also choose $0 < \rho_2 < \rho_1$ and consider $\hat{\xi} \mid \hat{\mathfrak{V}}(\rho_2) = \hat{\xi}_1 \in Z^l(\hat{\mathfrak{V}}(\rho_2), (q\mathcal{O})_X)$. Let us write $\hat{\xi}_1 = \sum \xi^*_{(\nu)}(t/\rho_2)^\nu$. Obviously we get $\xi^*_{(\nu)} = (\rho_2/\rho_1)^\nu \xi_{(\nu)} \mid \mathfrak{V}$. It follows easily that $\sup_\nu \|\xi^*_{(r)}\| < \infty$.

The previous lemma shows that $\overset{*}{\xi}_{(v)} = \sum a_{v\lambda}\, \mathfrak{b}_\lambda + \delta\eta_v$ where $\eta_v \in C^{l-1}(\mathfrak{B})$ with $\| \eta_v \mid \mathfrak{B}_1 \| \leqslant K \| \overset{*}{\xi}_{(v)} \|$ and $| a_{v\lambda} | \leqslant K \| \overset{*}{\xi}_{(v)} \|$. Let us put $a_\lambda = \sum a_{v\lambda} (t/\rho_2)^v$ and $\hat{\eta} = \sum \eta_v (t/\rho_2)^v$. We see that $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}_1(\rho_2))$ and $a_\lambda \in I(E^n(\rho_2))$. An easy computation gives $\hat{\xi}_1 \mid \hat{\mathfrak{B}}_1(\rho_2) = \sum a_\lambda \hat{\mathfrak{b}}_\lambda \mid \hat{\mathfrak{B}}_1(\rho_2) + \delta\,\hat{\eta}$. It follows by definition that $\xi_0 = \sum a_\lambda \hat{\mathfrak{b}}_\lambda$. We have now proved that $\hat{\mathfrak{b}}_1 \dots \hat{\mathfrak{b}}_r$ generate $\psi_{(l)}((q\mathcal{O})_X)$ at the origin. It follows in the same way that $\hat{\mathfrak{b}}_1 \dots \hat{\mathfrak{b}}_r$ generate $\psi_{(l)}((q\mathcal{O})_X)$ for every $t \in E^n(\rho_0)$ because it is enough to do everything in a polydisc around t. Now we also prove that the sheaf $\psi_{(l)}((q\mathcal{O})_X)$ is free, i.e. there are no relations between $\hat{\mathfrak{b}}_1 \dots \hat{\mathfrak{b}}_r$ at any point. Say for example that $a_1 \hat{\mathfrak{b}}_1 + \dots + a_r \hat{\mathfrak{b}}_r = 0$ at $\psi_{(l)}((q\mathcal{O})_X)_{(0)}$ where a_ι are germs of analytic functions at the origin in $E^n(\rho_0)$. Hence $\tilde{a}_1 \hat{\mathfrak{b}}_1 + \dots + \tilde{a}_r \hat{\mathfrak{b}}_r = 0$ in $H^l(X(\rho), (q\mathcal{O})_X)$ for some $\rho > 0$ with $\tilde{a}_\iota \in I(E^n(\rho))$. It follows that $\sum \tilde{a}_v \hat{\mathfrak{b}}_v = \delta\,\hat{\xi}$ in $X(\rho)$ for some $\hat{\xi} \in C^{l-1}(\hat{\mathfrak{U}}(\rho), (q\mathcal{O})_X)$. Take a point $t \in E^n(\rho)$ where some $\tilde{a}_v \neq 0$. Now we see that on $\{t\} \times X_0$ we have $\tilde{a}_1(t)\,\mathfrak{b}_1 + \dots + \tilde{a}_r(t)\,\mathfrak{b}_r = \partial\,\hat{\xi} \mid \{t\} \times X_0 \in C^{l-1}(\mathfrak{U}, (q\mathcal{O})_{X_0})$. This gives a contradiction to the fact that $\mathfrak{b}_1 \dots \mathfrak{b}_r$ are a base of $H^l(X_0, (q\mathcal{O})_{X_0})$.

Measure charts

Let X be a connected complex analytic manifold of dimension m. Let F be a holomorphic vector bundle of rank q on X and **F** the sheaf of holomorphic crossections in F. This sheaf is locally free. A regular proper holomorphic map $\psi: X \to E^n$ is given. Let us put $X_0 = \psi^{-1}(0)$. Now X_0 is a compact analytic manifold of dimension $m - n$. We now introduce special open coverings around X_0 in X.

Definition. A measure chart $\mathscr{W} = (\hat{W}, \Phi, \Theta, \rho)$ is a quadruple satisfying the conditions:

1) $\hat{W} \subset X$ is open and $W = \hat{W} \cap X_0$ is Stein.

2) $\Phi: \hat{W} \to E^n(\rho) \times W$ is a biholomorphic map such that the following diagram is commutative: