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Autor: Grauert, H.
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The previous lemma shows that $\xi_{(v)}^* = \sum a_{v\lambda} b_\lambda + \delta \eta_v$ where $\eta_v \in C^{l-1}(\mathfrak{B})$ with $\|\eta_v|_{\mathfrak{B}_1}\| \leq K \|\xi_{(v)}^*\|$ and $|a_{v\lambda}| \leq K \|\xi_{(v)}^*\|$. Let us put $a_\lambda = \sum a_{v\lambda} (t/\rho_2)^v$ and $\hat{\eta} = \sum \eta_v (t/\rho_2)^v$. We see that $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}_1(\rho_2))$ and $a_\lambda \in I(E^n(\rho_2))$. An easy computation gives $\hat{\xi}_1|_{\hat{\mathfrak{B}}_1(\rho_2)} = \sum a_\lambda \hat{b}_\lambda|_{\hat{\mathfrak{B}}_1(\rho_2)} + \delta \hat{\eta}$. It follows by definition that $\xi_0 = \sum a_\lambda \hat{b}_\lambda$. We have now proved that $\hat{b}_1 \dots \hat{b}_r$ generate $\psi_{(t)}((q\mathcal{O})_X)$ at the origin. It follows in the same way that $\hat{b}_1 \dots \hat{b}_r$ generate $\psi_{(t)}((q\mathcal{O})_X)$ for every $t \in E^n(\rho_0)$ because it is enough to do everything in a polydisc around t . Now we also prove that the sheaf $\psi_{(t)}((q\mathcal{O})_X)$ is free, i.e. there are no relations between $\hat{b}_1 \dots \hat{b}_r$ at any point. Say for example that $a_1 \hat{b}_1 + \dots + a_r \hat{b}_r = 0$ at $\psi_{(t)}((q\mathcal{O})_X)_{(0)}$ where a_i are germs of analytic functions at the origin in $E^n(\rho_0)$. Hence $\tilde{a}_1 \hat{b}_1 + \dots + \tilde{a}_r \hat{b}_r = 0$ in $H^l(X(\rho), (q\mathcal{O})_X)$ for some $\rho > 0$ with $\tilde{a}_i \in I(E^n(\rho))$. It follows that $\sum \tilde{a}_v \hat{b}_v = \delta \hat{\xi}$ in $X(\rho)$ for some $\hat{\xi} \in C^{l-1}(\hat{\mathcal{U}}(\rho), (q\mathcal{O})_X)$. Take a point $t \in E^n(\rho)$ where some $\tilde{a}_v \neq 0$. Now we see that on $\{t\} \times X_0$ we have $\tilde{a}_1(t) b_1 + \dots + \tilde{a}_r(t) b_r = \delta \hat{\xi}|_{\{t\} \times X_0} \in C^{l-1}(\mathcal{U}, (q\mathcal{O})_{X_0})$. This gives a contradiction to the fact that $b_1 \dots b_r$ are a base of $H^l(X_0, (q\mathcal{O})_{X_0})$.

MEASURE CHARTS

Let X be a connected complex analytic manifold of dimension m . Let F be a holomorphic vector bundle of rank q on X and \mathbf{F} the sheaf of holomorphic crosssections in F . This sheaf is locally free. A regular proper holomorphic map $\psi: X \rightarrow E^n$ is given. Let us put $X_0 = \psi^{-1}(0)$. Now X_0 is a compact analytic manifold of dimension $m - n$. We now introduce special open coverings around X_0 in X .

Definition. A measure chart $\mathcal{W} = (\hat{W}, \Phi, \Theta, \rho)$ is a quadruple satisfying the conditions:

- 1) $\hat{W} \subset X$ is open and $W = \hat{W} \cap X_0$ is Stein.
- 2) $\Phi: \hat{W} \rightarrow E^n(\rho) \times W$ is a biholomorphic map such that the following diagram is commutative:

$$\begin{array}{ccc} \hat{W} & \xrightarrow{\Phi} & E^n(\rho) \times W \\ \psi \searrow & & \swarrow \pi \\ & & E^n(\rho). \end{array}$$

Here π is the projection map.

3) $\Theta: F|_{\hat{W}} \rightarrow \hat{W} \times \mathbf{C}^q$ is a trivialization of F on \hat{W} .

If \mathcal{W} is a given measure chart on X we can identify the sheaf $(\hat{W}, F|_{\hat{W}})$ of \mathcal{C}_X -modules with the sheaf $(W \times E^n(\rho), q\mathcal{O})$ using Φ and Θ . If $U \subset W$ is open and $\rho' \leq \rho$ we put $\hat{U}(\rho') = \Phi^{-1}(U \times E^n(\rho'))$. Hence if $\hat{s} \in \Gamma(\hat{U}(\rho'), F)$ we can identify \hat{s} with an element of $\Gamma(U \times E^n(\rho'), q\mathcal{O})$. We shall simply denote this element of $\Gamma(U \times E^n(\rho'), q\mathcal{O})$ by the same letter \hat{s} . Now we can expand \hat{s} in a Taylor series: $\hat{s} = \sum_{|v|=0}^{\infty} s_v (t/\rho')^v$ where $s_v \in qI(U)$.

Definition of a norm. When $\hat{s} \in \Gamma(\hat{U}(\rho'), F)$ we put $\|\hat{s}\| = \sup_v |s_v(U)|$.

Strictly speaking the norm $\|\hat{s}\|$ is taken with respect to the measure chart \mathcal{W} .

It is not hard to see that for every point $x \in X_0$ there exists a measure chart \mathcal{W} such that $x \in \hat{W}$. In particular we can cover X_0 by finitely many measure charts $\mathcal{W}_\iota = (\hat{W}_\iota, \Phi_\iota, \Theta_\iota, \rho_\iota)$, i.e. $X_0 \subset \subset \bigcup_1^{\iota^*} \hat{W}_\iota$. We remark that it

follows that $X(\rho) = \psi^{-1}(E^n(\rho)) \subset \subset \bigcup_1^{\iota^*} \hat{W}_\iota$ for some $\rho > 0$ with $\rho \leq \rho_\iota$ because ψ is a proper map. The collection $\mathcal{W} = \{\mathcal{W}_\iota\}_1^{\iota^*}$ is called an atlas around X_0 . From now on \mathcal{W} is a fixed atlas.

Measure coverings. We shall define measure coverings with respect to the given atlas \mathcal{W} above. If $U \subset W_\iota$ is open we put $(U)_\iota(\rho) = \Phi_\iota^{-1}(U \times E^n(\rho))$ when $\rho \leq \rho_\iota$. We see that $(U)_\iota(\rho) \subset \hat{W}_\iota$ and $(U)_\iota(\rho)$ is Stein if U is Stein. Let $\mathfrak{U} = \{U_\iota\}_1^{\iota^*}$ be a Stein covering of X_0 with $U_\iota \subset \subset W_\iota$ for each ι . Let $\rho > 0$ with $\rho < \min_\iota \rho_\iota$. We put $\hat{U}_\iota(\rho) = (U_\iota)_\iota(\rho)$. We see that $\hat{U}_\iota(\rho) \subset \subset \hat{W}_\iota$ and $\hat{U}_\iota(\rho)$ are Stein. It is now required that $\hat{\mathfrak{U}}(\rho) =$

$= \{ \hat{U}_i(\rho) \}_1^{i^*}$ is a Stein covering of $X(\rho)$. We say then that $\hat{\mathcal{U}}(\rho)$ is a measure covering of $X(\rho)$.

Admissible refinements of measure coverings. Let $\hat{\mathcal{U}}(\rho)$ and $\hat{\mathcal{U}}^*(\rho)$ be two measure coverings of $X(\rho)$. We say that $\hat{\mathcal{U}}^*(\rho)$ is an admissible refinement of $\hat{\mathcal{U}}(\rho)$ if the following conditions hold:

- 1) $U_i^* \subset \subset U_i$ for each i .
- 2) If $U_{i_0 \dots i_\lambda}^* = U_{i_0}^* \cap \dots \cap U_{i_\lambda}^*$ we put $(U_{i_0 \dots i_\lambda}^*)_v = \Phi_v^{-1}(U_{i_0 \dots i_\lambda}^* \times E^n(\rho))$ for each $v \in \{i_0 \dots i_\lambda\}$. It is now required that $(U_{i_0 \dots i_\lambda}^*)_v \subset (U_{i_0 \dots i_\lambda}^*)_\mu$ for all $v, \mu \in \{i_0 \dots i_\lambda\}$.
- 3) $\hat{U}_{i_0 \dots i_\lambda}^* = \hat{U}_{i_0}^* \cap \dots \cap \hat{U}_{i_\lambda}^* \subset (U_{i_0 \dots i_\lambda}^*)_\mu$ for each $\mu \in \{i_0 \dots i_\lambda\}$.

EXISTENCE OF ADMISSIBLE REFINEMENTS OF MEASURE COVERINGS

Existence Theorem. For every fixed integer s we can find, for some $\rho > 0$, a sequence $\mathcal{U}_s \ll \mathcal{U}_{s-1} \ll \dots \ll \mathcal{U}_1 \ll \mathcal{U}_0$ of finer measure coverings of $X(\rho)$ each of which is an admissible refinement of the following.

Proof. We first construct a measure covering of $X(\rho)$ for some $\rho < \min \rho_i$. Let $\mathcal{U}_0 = \{ \mathcal{U}_i \}_1^{i^*}$ be a Stein covering of X_0 such that $U_i \subset \subset W_i$ for $i \in \{1, \dots, i^*\}$. Choose a fixed $\rho_0 < \min \rho_i$. Now the open sets $\Phi_i^{-1}(U_i \times E^n(\rho_0))$ cover X_0 and hence they also cover $X(\rho)$ for some sufficiently small ρ . Hence \mathcal{U}_0 defines a measure covering of $X(\rho)$. It is also clear that \mathcal{U}_0 defines a measure covering of $X(\rho')$ for each $\rho' \leq \rho$. Let us now construct \mathcal{U}_1 . We let $\mathcal{U}^* = \{ U_i^* \}_1^{i^*}$ be a Stein covering such that $U_i^* \subset \subset U_i$ always holds. Now we can find $\rho_1 \leq \rho$ such that $\{ \hat{U}_i^*(\rho_1) = \Phi_i^{-1}(U_i^* \times E^n(\rho_1)) \}_1^{i^*}$ cover $X(\rho_1)$. Hence $\hat{\mathcal{U}}^*(\rho_1)$ and $\hat{\mathcal{U}}(\rho_1)$ are measure coverings of $X(\rho_1)$. But we do not yet know if $\hat{\mathcal{U}}^*(\rho_1) \ll \hat{\mathcal{U}}(\rho_1)$. We claim that if $\rho_2 \leq \rho_1$ is sufficiently small then $\hat{\mathcal{U}}^*(\rho_2) \ll \hat{\mathcal{U}}(\rho_2)$. For suppose this is false. Say that 2) fails for $\hat{\mathcal{U}}^*(\rho_2)$ and $\hat{\mathcal{U}}(\rho_2)$ when $0 < \rho_2 \leq \rho_1$. Hence $\Phi_v^{-1}(U_{i_0 \dots i_\lambda}^* \times E^n(\rho_2)) - \Phi_\mu^{-1}(U_{i_0 \dots i_\lambda} \times E^n(\rho_2))$ are non empty for suitable indices while $\rho_2 \rightarrow 0$. Choose a point x_t from each of these sets. Because $x_t \in X(\rho_1)$ which is relatively compact we may assume that $x_t \rightarrow x_0$. Obviously we get $x_0 \in \overline{U_{i_0 \dots i_\lambda}^*} - U_{i_0 \dots i_\lambda}$, a contradic-