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tion because $\overline{U_{\iota_0 \dots \iota_\lambda}^*} \subset \overline{U_{\iota_0}^*} \cap \dots \cap \overline{U_{\iota_\lambda}^*} \subset U_{\iota_0 \dots \iota_\lambda}$. In the same way we can prove that condition 3) is satisfied if ρ_2 is sufficiently small and the theorem is clear.

GENERAL THEORY

Let G be an analytic manifold. We put $\hat{G} = G \times E^n(\rho_1)$ where ρ_1 is an n -tuple of positive numbers. Let $\pi: \hat{G} \rightarrow E^n(\rho_1)$ and $\mathfrak{P}: \hat{G} \rightarrow G$ be the projection maps. $\hat{G}^* \subset \hat{G}$ denotes an open subset and $G^* = \hat{G}^* \cap G \times \{0\}$.

The set G^* can be identified with an open subset of G . We denote by $\alpha: G^* \times E^n(\rho_1) \rightarrow \hat{G}^*$ a biholomorphic fiber preserving map, i.e. $\pi \circ \alpha = \pi^*$ where $\pi^*: G^* \times E^n(\rho_1) \rightarrow E^n(\rho_1)$ is the natural projection. Let $\rho \leq \rho_2 = \gamma \rho_1 < \rho_1$ where $0 < \gamma < 1$ is a fixed number. We put $\hat{G}(\rho) = G \times E^n(\rho)$. If f is a holomorphic function on $\hat{G}(\rho)$ we write $f = \sum a_\nu (t/\rho)^\nu$ with $a_\nu \in I(G)$. We define the norm $\|f\|_\rho$ of f by $\|f\|_\rho = \sup_\nu \{ \sup |a_\nu(G)| \}$.

If $f \in I(\hat{G}(\rho))$ we see that $f \circ \alpha$ is a well defined function on $G^* \times E^n(\rho)$ because α is fiber preserving. We define $\|f \circ \alpha\|_\rho$ using G^* instead of G as above. We have the proposition:

Proposition 1. There exists a constant K such that $\|f \circ \alpha\|_\rho \leq K \|f\|_\rho$ where $K = K(\rho_2)$ is independent of $\rho \leq \rho_2$.

Proof. We write $f = \sum_{|\nu|=0}^{\infty} a_\nu (t/\rho)^\nu$ with $a_\nu \in I(G)$. Now we get $f \circ \alpha = \sum (a_\nu \circ \mathfrak{P} \circ \alpha) (t/\rho)^\nu$ because α is fiber preserving. Since $\mathfrak{P}(\hat{G}^*) \subset G$ we get $|a_\nu \circ \mathfrak{P}(\hat{G}^*)| \leq |a_\nu(G)| \leq \|f\|_\rho$. Now $a_\nu \circ \mathfrak{P} \circ \alpha$ admits a Taylor series: $a_\nu \circ \mathfrak{P} \circ \alpha = \sum C_{\nu\lambda} (t/\rho)^\lambda$ with $C_{\nu\lambda} \in I(G^*)$. Since $|\sum C_{\nu\lambda} (t/\rho)^\lambda| \leq \|f\|_\rho$ in $G^* \times E^n(\rho_1)$ and $\rho \leq \rho_2 = \gamma \rho_1$ Cauchy's inequalities give us $|C_{\nu\lambda}(G^*)| \leq \|f\|_\rho \gamma^{|\lambda|}$. Let us put $b_\mu = \sum_{\nu+\lambda=\mu} C_{\nu\lambda}$. We get $|b_\mu(G^*)| \leq \|f\|_\rho \sum \gamma^{|\lambda|} = \|f\|_\rho (1-\gamma)^{-n} = K \|f\|_\rho$. Now we can write $f \circ \alpha = \sum_\nu a_\nu \circ \mathfrak{P} \circ \alpha (t/\rho)^\nu = \sum_{\lambda,\nu} C_{\nu\lambda} (t/\rho)^\lambda (t/\rho)^\nu = \sum_\mu b_\mu (t/\rho)^\mu$. By definition we have $\|f \circ \alpha\|_\rho = \sup_\mu |b_\mu(G^*)| \leq K \|f\|_\rho$.

Let us now consider $\mathbf{h} = (h_{\nu\mu})$ which is a $q \times q$ matrix with $h_{\nu\mu} \in I(\hat{G})$. The $h_{\nu\mu}$ are also assumed to be bounded on \hat{G} .

Proposition 2. Let $\mathbf{f} = (f_1 \dots f_q) \in qI(\hat{G}(\rho))$. Then $\|\mathbf{h}(\mathbf{f})\|_\rho \leq K \|\mathbf{f}\|_\rho$. As before $\rho \leq \rho_2 = \gamma\rho_1 < \rho_1$ and K only depends on ρ_2 .

Proof. We have $\mathbf{h}(\mathbf{f}) = (g_1 \dots g_q)$ with $g_v = \sum_\mu h_{v\mu} f_\mu$. Let us write $h_{v\mu} = \sum_\lambda a_{v\mu\lambda} (t/\rho)^\lambda$. By assumption $|h_{v\mu}(\hat{G})| \leq M$ for some constant M and hence we have, by Cauchy's inequalities, $|a_{v\mu\lambda}(G)| \leq M\gamma^{|\lambda|}$. Let us also write $f_\mu = \sum_\lambda b_{\mu\lambda} (t/\rho)^\lambda$. By definition $\sup_{\mu,\lambda} |b_{\mu\lambda}(G)| = \|\mathbf{f}\|_\rho$. Now we get $g_v = \sum_\mu \sum_{\lambda_1, \lambda_2} a_{v\mu\lambda_1} b_{\mu\lambda_2} (t/\rho)^{\lambda_1 + \lambda_2} = \sum_{\mu, \lambda} C_{v\lambda} (t/\rho)^\lambda$ where $C_{v\lambda} = \sum_{\mu} \sum_{\lambda_1 + \lambda_2 = \lambda} a_{v\mu\lambda_1} b_{\mu\lambda_2}$. We get easily $|C_{v\lambda}(G)| \leq qM \|\mathbf{f}\|_\rho (1-\gamma)^{-n} = K \|\mathbf{f}\|_\rho$. Hence $\|\mathbf{h}(\mathbf{f})\|_\rho = \sup_v \|g_v\|_\rho = \sup_{v,\lambda} |C_{v\lambda}(G)| \leq K \|\mathbf{f}\|_\rho$.

We shall now apply these two propositions to our situation. Let $G^* \subset G \subset W_{\iota_0 \dots \iota_\lambda} \subset X_0$. Here G^* and G are open sets and $W_{\iota_0 \dots \iota_\lambda}$ comes from the measure atlas \mathcal{W} . As before $\rho \leq \rho_2 < \rho_* = \min \rho_{\iota_i}$. We are given ι and ι' from $\{\iota_0, \dots, \iota_\lambda\}$ and the following inclusions are assumed: $(G^*)_{\iota'}(\rho_1) \subset (G)_\iota(\rho_1)$, $(G^*)_{\iota'}(\rho_1) \subset \subset \hat{W}_{\iota'}$, $(G)_\iota(\rho_1) \subset \subset \hat{W}_\iota$.

The following theorem is very important.

Theorem I. Let $S \in \Gamma((G)_\iota(\rho), \mathbf{F})$. Then $\|S|(G^*)_{\iota'}(\rho)\|_{\iota'} \leq K \|S\|_\iota$. K depends only on ρ_2 .

Proof. We have the following diagram:

$$\begin{array}{ccc} & \Phi_\iota & \\ & (G)_\iota(\rho_1) \rightarrow G \times E^n(\rho_1) & \\ \text{injection} \uparrow & & \uparrow \alpha \\ & \Phi_{\iota'} & \\ & (G^*)_{\iota'}(\rho_1) \rightarrow G^* \times E^n(\rho_1) & \end{array}$$

α being a fiber preserving holomorphic map. We identify $S|(G^*)_{\iota'}(\rho)$ with an element of $qI(G^* \times E^n(\rho))$ using the trivialization of F in the chart $\mathcal{W}_{\iota'}$. Call this element S^* . Also S itself is considered as an element of $qI(G \times E^n(\rho))$ using the trivialization in the chart \mathcal{W}_ι . Now we have $S^* = \mathbf{h}(S \circ \alpha)$ where \mathbf{h} is a $q \times q$ matrix. The elements of \mathbf{h} are holomorphic functions defined on $\Phi_{\iota'}(\hat{W}_{\iota'}) \supset \supset G^* \times E^n(\rho_1)$. Hence the elements of \mathbf{h} are bounded on $G^* \times E^n(\rho_1)$. It is now obvious how we can use 1) and 2) to finish the proof.

We shall need one more general result. Let G be an analytic manifold. G is assumed to be Stein and $R^* = \{U_1, \dots, U_{\iota^*}\}$ a Stein covering of G .

The set $G^* \subset G$ is open and $R^{**} = \{V_1, \dots, V_{l^*}\}$ an open covering of G^* such that $V_l \subset \subset U_l$ for $l \in \{1, \dots, l^*\}$. We have:

Cartan's Theorem. There exists a constant K such that if $\xi \in Z^l(R^*, q\mathcal{O})$ then $\xi|_{R^{**}} = \delta\eta$ where $\eta \in C^{l-1}(R^{**}, q\mathcal{O})$ and $\|\eta\| \leq K\|\xi\|$ for $l \geq 1$.

This is a simple consequence of Theorem B and Banach's open mapping theorem.

Now we apply Cartan's theorem. We keep the notations as above. Let $\hat{G} = G \times E^n(\rho)$ and put $\hat{R}^* = \{U_l \times E^n(\rho)\}$. Now \hat{R}^* is a Stein covering of \hat{G} . Let $\hat{G}^* = G^* \times E^n(\rho)$ and $\hat{R}^{**} = \{V_l \times E^n(\rho)\}$. Let $\hat{\xi} \in Z^l(\hat{R}^*, q\mathcal{O})$ and write $\hat{\xi} = \sum \xi_{(v)}(t/\rho)^v$ with $\xi_{(v)} \in Z^l(R^*, q\mathcal{O})$. We assume $\|\hat{\xi}\|_\rho = \sup_v \|\xi_{(v)}\| < \infty$. Now Cartan's theorem gives $\xi_{(v)}|_{R^{**}} = \delta\eta_v$ with $\eta_v \in C^{l-1}(R^{**}, q\mathcal{O})$ and $\|\eta_v\| \leq K\|\xi_{(v)}\| < \infty$. It follows that $\hat{\eta} = \sum \eta_v(t/\rho)^v$ is well defined in $C^{l-1}(\hat{R}^{**}, q\mathcal{O})$ and by definition we have $\|\hat{\eta}\|_\rho \leq K\|\hat{\xi}\|_\rho$.

SMOOTHING

We are given a sequence of admissible refinements of measure coverings in $X(\rho_1)$. Here $\rho_1 < \rho_0 = \min \rho_l$ as usual. Let l be a fixed integer ≥ 1 . We are given $\mathfrak{B}^* \ll \mathfrak{B}' = \mathfrak{B}_{3l} \ll \mathfrak{B}_{3l-1} \ll \dots \ll \mathfrak{B}_1 \ll \mathfrak{B}_0 \ll \mathfrak{B} \ll \mathfrak{U}^* \ll \mathfrak{U} = \mathfrak{U}_{3l} \ll \dots \ll \mathfrak{U}_0 \ll \mathfrak{U}'$. Here it is also required that $(\mathfrak{B}_{v+1}, \mathfrak{U}_{v+1}) \ll (\mathfrak{B}_v, \mathfrak{U}_v); (\mathfrak{B}^*, \mathfrak{U}^*) \ll (\mathfrak{B}', \mathfrak{U})$ and $(\mathfrak{B}_0, \mathfrak{U}_0) \ll (\mathfrak{B}, \mathfrak{U}')$. These extra conditions mean: 1) $\hat{U}_{i_0 \dots i_\kappa}^{(v+1)} \cap \hat{V}_{l_0 \dots l_l}^{(v+1)} \subset (U_{i_0 \dots i_\kappa}^{(v)} \cap V_{l_0 \dots l_l}^{(v)})_i$ for each $i \in \{i_0, \dots, i_\kappa\}$ and 2) $(U_{i_0 \dots i_\kappa}^{(v+1)} \cap V_{l_0 \dots l_l}^{(v+1)})_j \subset (U_{i_0 \dots i_\kappa}^{(v)} \cap V_{l_0 \dots l_l}^{(v)})_i$ for all $i, j \in \{i_0, \dots, i_\kappa, l_0, \dots, l_l\}$.

Recall that all operations are done with respect to ρ_1 . Let us put $\hat{R}_{i_0 \dots i_\kappa l_0 \dots l_\kappa}^{(v)} = \hat{U}_{i_0 \dots i_\kappa}^{(v)} \cap \hat{V}_{l_0 \dots l_\kappa}^{(v)}$. We consider elements $\xi_{i_0 \dots i_\kappa l_0 \dots l_\kappa} \in \hat{\Gamma}(\hat{R}_{i_0 \dots i_\kappa l_0 \dots l_\kappa}^{(v)}, \mathbb{F})$.

Now we take a full collection $\hat{\xi} = \{\hat{\xi}_{i_0 \dots i_\kappa l_0 \dots l_\kappa}\}$ of such elements which is anticommutative in $\{i_0, \dots, i_\kappa\}$ and $\{l_0, \dots, l_\kappa\}$. In this way we get a double complex $C_v^{k, \kappa}$. Here $\delta : C_v^{k, \kappa} \rightarrow C_v^{k+1, \kappa}$ and $\partial : C_v^{k, \kappa} \rightarrow C_v^{k, \kappa+1}$ are the usual coboundary operators.

NORM IN $C_v^{k, \kappa}$: Let $\hat{\xi} \in C_v^{k, \kappa}$; we put