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APPROXIMATION

We use positive n -tuples ρ, \dots with $\rho \leq \rho_2 < \rho_3 < \rho_4 < \rho_1$ and $\rho = \gamma'' \rho_1, \rho_2 = \gamma \rho_1, \rho_3 = \gamma' \rho_1, \rho_4 = \gamma''' \rho_1$. The n -tuple ρ_1 is defined as in the smoothing theorem.

Definition: $H_*^l = \{ \xi \in H^l(X_0, \underline{F}|X_0) \text{ such that there exists } U = U(0) \text{ in } E^n \text{ with } \hat{\xi} \in H^l(\psi^{-1}(U), \mathbf{F}) \text{ and } \hat{\xi}|X_0 = \xi \}$. Serre's theorem gives $\dim_{\mathbf{C}} H_*^l \leq \dim_{\mathbf{C}} H^l(X_0, \underline{F}|X_0) < \infty$. In the following discussion we are given $\hat{b}_1, \dots, \hat{b}_r$ in $Z^l(\hat{\mathcal{U}}'(\rho_4), \mathbf{F})$ such that $\hat{b}_1|X_0, \dots, \hat{b}_r|X_0$ constitute a base of the complex vector space H_*^l . For this to be possible, ρ_4 has to be chosen small enough. Here $\hat{\mathcal{U}}'$ is a Stein covering of $X(\rho_1)$ and defined as in the smoothing theorem. We also assume that we are given a sequence of measure coverings as there. Further we construct the sequence so that there are still sufficiently many measure coverings in between \mathfrak{B} and \mathcal{U} . These are denoted by \mathcal{U}_v^* . We have $\mathcal{U} \gg \mathcal{U}_1^* \gg \mathcal{U}_2^* \gg \dots \gg \mathfrak{B}$. The n -tuple ρ_3 is also fixed from now on and K always denotes (possibly different) constants.

Approximation Lemma: Let $\varepsilon > 0$. Then we can find ρ_2 such that: If $\rho \leq \rho_2$ and $\hat{\xi} \in Z^l(\hat{\mathcal{U}}(\rho), \mathbf{F})$ with $\|\hat{\xi}\|_{\rho} < \infty$ (the norm is taken with respect to $\hat{\mathcal{U}}_1^*(\rho)$), then there exist $a_1, \dots, a_r \in I(E^n(\rho))$ and $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ such that $\tilde{\xi} = \hat{\xi} - \sum_1^r a_i \hat{b}_i - \delta \hat{\eta}$ on $\hat{\mathfrak{B}}(\rho)$. Here $\tilde{\xi} \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ and $\|\tilde{\xi}\|_{\rho} \leq \varepsilon \|\hat{\xi}\|_{\rho}$ and $\|a_v\|_{\rho}, \|\hat{\eta}\|_{\rho} \leq K \|\hat{\xi}\|_{\rho}$. K is a fixed constant.

Proof. We shall first prove some results which are needed later on. Let $S \in \Gamma(\hat{U}_{\iota_0 \dots \iota_l}(\rho), \mathbf{F})$. Choose $\iota \in \{ \iota_0, \dots, \iota_l \}$. Now $(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota} \subset \hat{U}_{\iota_0 \dots \iota_l}$ because $\mathcal{U}_1^* \ll \mathcal{U}$. The operations are always defined with respect to ρ_1 . We can now restrict S to $(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota}(\rho)$. In the chart \mathcal{W}_{ι} we can write $S = \sum a_v (t/\rho)^v$. Here $a_v \in qI(U_{\iota_0 \dots \iota_l}^{(1)*})$. Now the a_v are extended constantly and we get elements $\hat{a}_v \in \Gamma((U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota}, \mathbf{F})$. Let us put $S_v = \hat{a}_v | \hat{U}_{\iota_0 \dots \iota_l}^{(2)*}$. We claim that $\|S_v\|_{\rho_1} \leq K \|S\|_{\rho}$. For obviously $\|S\|_{\rho} \geq |a_v (U_{\iota_0 \dots \iota_l}^{(1)*})|$ and

we can use the Theorem I to prove that $\|S_v\|_{\rho_1} \leq K \|\hat{a}_v\| (U_{\iota_0 \dots \iota_l}^{(1)*}(\rho_1))\|_{\iota} = K \|a_v(U_{\iota_0 \dots \iota_l}^{(1)*})\| \leq K \|S\|_{\rho}$. Q.E.D.

Let S'_v be defined using some other $\iota' \in \{\iota_0, \dots, \iota_l\}$. Then $S_v - S'_v \in \Gamma(\hat{U}_{\iota_0 \dots \iota_l}^{(2)*}, \mathbf{F})$. We claim that $\|S_v - S'_v\|_{\rho_4} \leq K \gamma''' \|S\|_{\rho}$.

Proof. Define $\alpha_s = \sum_{|\lambda|=s} a_{\lambda}(t/\rho)^{\lambda}$ and $\beta_s = \sum_{|\lambda|=0}^{s-1} a_{\lambda}(t/\rho)^{\lambda}$ over $(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota}(\rho)$. We do the same for ι' respectively and obtain α'_s and β'_s over $(U_{\iota_0 \dots \iota_l}^{(1)*})_{\iota'}(\rho)$. For the restrictions to $\hat{U}_{\iota_0 \dots \iota_l}^{(2)*}$ we see that $\alpha_s - \alpha'_s = -(\beta_s - \beta'_s)$. Hence we get $\|\alpha_s - \alpha'_s\|_{\rho_4} \leq K(\gamma''')^s \|\alpha_s - \alpha'_s\|_{\rho_1} = K(\gamma''')^s \|\beta_s - \beta'_s\|_{\rho_1} \leq K(\gamma''')^s \|\beta_s\|_{\rho_1} + K(\gamma''')^s \|\beta'_s\|_{\rho_1} \leq K(\gamma''')^s [\|\beta_s\|_{\rho_1}^* + \|\beta'_s\|_{\rho_1}^*] \leq K(\gamma''')^s (\gamma'')^{1-s} \|S\|_{\rho}$. Here the norms are defined with respect to $U_{\iota_0 \dots \iota_l}^{(3)*}$ except $\|\cdot\|^*$ and $\|S\|_{\rho}$ which are defined with respect to $U_{\iota_0 \dots \iota_l}^{(1)*}$. Now we look at the difference $(S_v - S'_v) t^v/\rho^v$ on $(U_{\iota_0 \dots \iota_l}^{(3)*})_{\mu}$ with $|v|=s$, $\mu \in \{\iota_0, \dots, \iota_l\}$, and the power series development with respect to W_{μ} . There is one term of order s which is equal to the corresponding term of $\alpha_s - \alpha'_s$. Therefore its norm is $\leq K(\gamma''')^s \cdot (\gamma'')^{1-s} \|S\|_{\rho}$. Moreover we have $\|S_v(t/\rho)^v - S'_v(t/\rho)^v\|_{\rho_1} \leq (\gamma'')^{-s} \cdot K \|S\|_{\rho}$ where the first norm is defined with respect to $U_{\iota_0 \dots \iota_l}^{(3)*}$. For the sum \sum of terms of higher order than s in the power series of $(S_v - S'_v) t^v/\rho^v$ we therefore get: $\|\sum\|_{\rho_4} \leq (\gamma''')^{s+1} (\gamma'')^{-s} \cdot K \|S\|_{\rho}$. Hence we get $\|(S_v - S'_v)\|_{\rho_4} \leq \gamma''' \cdot K \|S\|_{\rho}$. This proves our statement. We see that K is independent of ρ_4 and S . The number γ''' depends on ρ_4 only, so $\gamma''' \cdot K$ gets very small if we make ρ_4 very small.

Let $\hat{\xi} \in Z^l(\hat{\mathcal{U}}(\rho), \mathbf{F})$ with $\hat{\xi} = \{\hat{\xi}_{\iota_0 \dots \iota_l}\}$. Choose $\iota = \iota(\iota_0, \dots, \iota_l)$ as a function of the unordered $(l+1)$ -tuple. We now fix ι_0, \dots, ι_l and write $S = \hat{\xi}_{\iota_0 \dots \iota_l}$. We apply to S the method described above and obtain $\hat{\xi}_{\iota_0 \dots \iota_l}^{(v)} = S_v$. We do this now for every ι_0, \dots, ι_l and consider $\hat{\xi}^{(v)} = \{\hat{\xi}_{\iota_0 \dots \iota_l}^{(v)}\}$ as an element of $C^l(\hat{\mathcal{U}}_2^*(\rho_4), \mathbf{F})$. Of course $\hat{\xi}^{(v)}$ depends on the choice of $\iota = \iota(\iota_0 \dots \iota_l)$ here. Now we see that $\|\hat{\xi}^{(v)}\|_{\rho_4} \leq \|\hat{\xi}^{(v)}\|_{\rho_1} \leq K \|\hat{\xi}\|_{\rho}$. We also wish to estimate $\delta \hat{\xi}^{(v)}$. Because $\hat{\xi} \in Z^l(\hat{\mathcal{U}}(\rho), \mathbf{F})$ we can use the preliminary result on ι and ι' to obtain $\|\delta \hat{\xi}^{(v)}\|_{\rho_4} \leq K \gamma''' \|\hat{\xi}\|_{\rho}$.

We shall also need another result:

Induction Lemma: There exists $\hat{\eta}_v \in C^l(\hat{\mathcal{U}}_4^*(\rho_3), \mathbf{F})$ such that $\delta\hat{\eta}_v = \delta\hat{\xi}_{(v)}$ on $\hat{\mathcal{U}}_4^*(\rho_3)$ and $\|\hat{\eta}_v\|_{\rho_3} \leq K \|\delta\hat{\xi}_{(v)}\|_{\rho_4}$.

Proof. The proof uses the assumption that $\psi_{(l+1)}(\mathbf{F})$ is coherent. Because the coherence of direct images is proved by downward induction on l , this assumption can be made. Moreover it is assumed that the main theorem is proved for dimension $l + 1$ already. Let us now put $\alpha = \delta\hat{\xi}_{(v)} \in B^{l+1}(\hat{\mathcal{U}}_2^*(\rho_4), \mathbf{F})$ and $\hat{\eta}_v = \beta \in C^l(\hat{\mathcal{U}}_4^*(\rho_3), \mathbf{F})$. We have to prove the existence of β . We may assume that ρ_4 is so small that the main theorem is valid for $\rho \leq \rho_4$ in the case of dimension $l + 1$. So there are cocycles $\omega_1, \dots, \omega_r \in Z^{l+1}(\hat{\mathcal{U}}(\rho_4), \mathbf{F})$ such that $\alpha = \sum C_\lambda \omega_\lambda + \delta\eta$, where $C_\lambda \in I(E^n(\rho_4))$ and $\eta \in C^l(\hat{\mathcal{U}}_4^*(\rho_4), \mathbf{F})$. We have to assume that between $\hat{\mathcal{U}}_4^*$ and \mathcal{U}_2^* there are very many measure coverings. The cross-sections $\psi_{(l+1)}(\omega_\lambda)$ give a homomorphism $r\mathcal{O} \rightarrow \psi_{(l+1)}(\mathbf{F})$ over $E^n(\rho_4)$. Because $\psi_{(l+1)}(\mathbf{F})$ is coherent the kernel \mathcal{N} is coherent again. Over $E^n(\rho')$ with $\rho_3 < \rho' < \rho_4$ we find an epimorphism $p\mathcal{O} \rightarrow \mathcal{N}$. Denote by n_1, \dots, n_p the images of the unit cross-sections in $p\mathcal{O}$. Write $n_\lambda = (e_{\lambda 1}, \dots, e_{\lambda r})$ as an r -tupel of holomorphic functions. The image of n_λ in $\Gamma(E^n(\rho'), \psi_{(l+1)}(\mathbf{F}))$ is $\psi_{(l+1)}(\sum_{\mu=1}^r e_{\lambda\mu} \omega_\mu)$ and zero. We may choose ρ_2 and then ρ_3 and ρ' very small. Then it follows that $\hat{n}_\lambda = \sum e_{\lambda\mu} \omega_\mu$ is a coboundary. If $\rho_3 < \rho'' < \rho'$ there are cochains $\eta_\lambda \in C^l(\hat{\mathcal{U}}_4^*(\rho''), \mathbf{F})$ such that $\delta\eta_\lambda = \hat{n}_\lambda$. Now $(C_1, \dots, C_r) \in \Gamma(E^n(\rho_4), \mathcal{N})$. By the methods of sheaf theory we can lift this cross-section to $p\mathcal{O}$. Using a "Banach open mapping theorem" we see that the map $\Gamma(E^n(\rho'), p\mathcal{O}) \rightarrow \Gamma(E^n(\rho'), \mathcal{N})$ is open. This means here that we can find holomorphic functions a_λ over $E^n(\rho_3)$ such that $C_\mu = \sum a_\lambda e_{\lambda\mu}$ and $\|a_\lambda\|_{\rho_3} \leq K \max_\mu \|C_\mu\|_{\rho'} \leq K \max_\mu \|C_\mu\|_{\rho_4}$. We get $\sum C_\mu \omega_\mu = \sum a_\lambda e_{\lambda\mu} \omega_\mu = \sum a_\lambda \hat{n}_\lambda = \delta(\sum a_\lambda \eta_\lambda)$. This leads to $\alpha \in C^{l+1}(\hat{\mathcal{U}}_4^*(\rho_3)) = \delta(\eta + \sum a_\lambda \eta_\lambda)$. The estimates required obviously hold. Q.E.D.

Let us now put $\hat{\xi}_{(v)}^* = \hat{\xi}_{(v)} - \hat{\eta}_v \in Z^l(\hat{\mathcal{U}}_4(\rho_3), \mathbf{F})$. We can write $\hat{\xi}_{(v)}^* \mid X_0 = \sum a_{v\lambda} \hat{b}_\lambda \mid X_0 + \delta\gamma_v$ over \mathcal{U}_6^* . Here $a_{v\lambda}$ are complex numbers and $\gamma_v \in C^{l-1}(\mathcal{U}_6^*, \mathbf{F} \mid X_0)$. Cartan's theorem and the result after that give the estimates $|a_{v\lambda}| \leq K \|\hat{\xi}_{(v)}^*\|_{\rho_3} \leq K \|\hat{\xi}\|_\rho$ and $\|\gamma_v\|_{\rho_3} \leq K \|\hat{\xi}_{(v)}^*\|_{\rho_3} \leq K \|\hat{\xi}\|_\rho$. Here $\hat{\gamma}_v \in C^{l-1}(\hat{\mathcal{U}}_7^*(\rho_3), \mathbf{F})$ has been obtained by a constant

extension of γ_v . Let us now put $\hat{\xi}_{(v)}^{(1)} = \hat{\xi}_{(v)}^* - \sum a_{v\lambda} \hat{b}_\lambda - \delta \hat{\gamma}_v$. Here $\hat{\xi}_{(v)}^{(1)} \in C^l(\hat{\mathfrak{U}}_7^*(\rho_3), \mathbb{F})$. Using the previous estimates and the fact that the \hat{b}_λ are finite we find that $\|\hat{\xi}_{(v)}^{(1)}\|_{\rho_3} \leq K \|\hat{\xi}_{(v)}\|_{\rho_4} \leq K \|\hat{\xi}\|_{\rho}$.

Now we also have $\hat{\xi}_{(v)}^{(1)}|_{X_0} = 0$. It follows that

$$\|\hat{\xi}_{(v)}^{(1)}\|_{\rho} \leq \gamma/\gamma' \|\hat{\xi}_{(v)}^{(1)}\|_{\rho_3} \leq \gamma/\gamma' \cdot K \|\hat{\xi}\|_{\rho}.$$

Finally we put in $\hat{\mathfrak{U}}_9^*(\rho)$:

$$\begin{aligned} \hat{\xi}^{(1)} &= \sum \hat{\xi}_{(v)}^{(1)} (t/\rho)^v = \\ &= \sum \hat{\xi}_{(v)} (t/\rho)^v - \sum \hat{\eta}_v (t/\rho)^v - \sum a_{v\lambda} (t/\rho)^v \hat{b}_\lambda - \delta (\sum \hat{\gamma}_v (t/\rho)^v) \\ &= \hat{\xi} - \hat{\eta} - \sum a_\lambda \hat{b}_\lambda - \delta \hat{\gamma}. \end{aligned}$$

Using the fact that the sum of the absolute values of the coefficients in the power series expansion of $\hat{\xi}_{(v)}^{(1)}$ by (t/ρ) is smaller than $\gamma/\gamma' \cdot K \|\hat{\xi}\|_{\rho}$ and that with respect to $\hat{\eta}_v$ is smaller than $\gamma''' \cdot K \|\hat{\xi}\|_{\rho}$ we find: $\|\hat{\xi}^{(1)}\|_{\rho} \leq \gamma/\gamma' \cdot K \|\hat{\xi}\|_{\rho}$ and $\|\hat{\eta}\|_{\rho} \leq \gamma''' \cdot K \|\hat{\xi}\|_{\rho}$ and $\|a_\lambda\|_{\rho} \leq K \|\hat{\xi}\|_{\rho}$. We take the restriction to $\hat{\mathfrak{B}}(\rho)$ and now $\tilde{\xi} = \hat{\xi}^{(1)} - \hat{\eta} \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbb{F})$ is the desired element. Of course we have to choose ρ_4 and then ρ_2 small enough, for example let $\gamma''' < \varepsilon/2 K$ and $\gamma \leq \varepsilon\gamma'/2 K$.

MAIN THEOREM

There exists ρ_2 and a constant K such that if $\rho \leq \rho_2$ and $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbb{F})$ with $\|\hat{\xi}\|_{\rho} < \infty$ then we can find $a_1, \dots, a_r \in I(E^n(\rho))$ and $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}(\rho), \mathbb{F})$ such that $\hat{\xi} = \sum a_\lambda \hat{b}_\lambda + \delta \hat{\eta}$ on $\hat{\mathfrak{B}}(\rho)$ with $\|\hat{\eta}\|_{\rho}$ and $\|a_v\|_{\rho} \leq K \|\hat{\xi}\|_{\rho}$.

Proof. We have one constant K from the smoothing theorem. Now we find ρ_2 with an ε in the Approximation Lemma such that $\varepsilon \cdot K < 1/2$. We shall use this ρ_2 and prove the theorem here. We are given $\hat{\xi}_0 = \hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbb{F})$ with $\|\hat{\xi}\|_{\rho} < \infty$. The Approximation Lemma gives $\tilde{\xi}_1 =$