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extension of  $\gamma_v$ . Let us now put  $\hat{\xi}_{(v)}^{(1)} = \hat{\xi}_{(v)}^* - \sum a_{v\lambda} \hat{b}_\lambda - \delta \hat{\gamma}_v$ . Here  $\hat{\xi}_{(v)}^{(1)} \in C^l(\hat{\mathfrak{U}}_7^*(\rho_3), \mathbb{F})$ . Using the previous estimates and the fact that the  $\hat{b}_\lambda$  are finite we find that  $\|\hat{\xi}_{(v)}^{(1)}\|_{\rho_3} \leq K \|\hat{\xi}_{(v)}\|_{\rho_4} \leq K \|\hat{\xi}\|_{\rho}$ .

Now we also have  $\hat{\xi}_{(v)}^{(1)}|_{X_0} = 0$ . It follows that

$$\|\hat{\xi}_{(v)}^{(1)}\|_{\rho} \leq \gamma/\gamma' \|\hat{\xi}_{(v)}^{(1)}\|_{\rho_3} \leq \gamma/\gamma' \cdot K \|\hat{\xi}\|_{\rho}.$$

Finally we put in  $\hat{\mathfrak{U}}_9^*(\rho)$ :

$$\begin{aligned} \hat{\xi}^{(1)} &= \sum \hat{\xi}_{(v)}^{(1)} (t/\rho)^v = \\ &= \sum \hat{\xi}_{(v)} (t/\rho)^v - \sum \hat{\eta}_v (t/\rho)^v - \sum a_{v\lambda} (t/\rho)^v \hat{b}_\lambda - \delta (\sum \hat{\gamma}_v (t/\rho)^v) \\ &= \hat{\xi} - \hat{\eta} - \sum a_\lambda \hat{b}_\lambda - \delta \hat{\gamma}. \end{aligned}$$

Using the fact that the sum of the absolute values of the coefficients in the power series expansion of  $\hat{\xi}_{(v)}^{(1)}$  by  $(t/\rho)$  is smaller than  $\gamma/\gamma' \cdot K \|\hat{\xi}\|_{\rho}$  and that with respect to  $\hat{\eta}_v$  is smaller than  $\gamma''' \cdot K \|\hat{\xi}\|_{\rho}$  we find:  $\|\hat{\xi}^{(1)}\|_{\rho} \leq \gamma/\gamma' \cdot K \|\hat{\xi}\|_{\rho}$  and  $\|\hat{\eta}\|_{\rho} \leq \gamma''' \cdot K \|\hat{\xi}\|_{\rho}$  and  $\|a_\lambda\|_{\rho} \leq K \|\hat{\xi}\|_{\rho}$ . We take the restriction to  $\hat{\mathfrak{B}}(\rho)$  and now  $\tilde{\xi} = \hat{\xi}^{(1)} - \hat{\eta} \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbb{F})$  is the desired element. Of course we have to choose  $\rho_4$  and then  $\rho_2$  small enough, for example let  $\gamma''' < \varepsilon/2 K$  and  $\gamma \leq \varepsilon\gamma'/2 K$ .

### MAIN THEOREM

There exists  $\rho_2$  and a constant  $K$  such that if  $\rho \leq \rho_2$  and  $\hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbb{F})$  with  $\|\hat{\xi}\|_{\rho} < \infty$  then we can find  $a_1, \dots, a_r \in I(E^n(\rho))$  and  $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}(\rho), \mathbb{F})$  such that  $\hat{\xi} = \sum a_\lambda \hat{b}_\lambda + \delta \hat{\eta}$  on  $\hat{\mathfrak{B}}(\rho)$  with  $\|\hat{\eta}\|_{\rho}$  and  $\|a_v\|_{\rho} \leq K \|\hat{\xi}\|_{\rho}$ .

*Proof.* We have one constant  $K$  from the smoothing theorem. Now we find  $\rho_2$  with an  $\varepsilon$  in the Approximation Lemma such that  $\varepsilon \cdot K < 1/2$ . We shall use this  $\rho_2$  and prove the theorem here. We are given  $\hat{\xi}_0 = \hat{\xi} \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbb{F})$  with  $\|\hat{\xi}\|_{\rho} < \infty$ . The Approximation Lemma gives  $\tilde{\xi}_1 =$

$= \hat{\xi} - \sum a_{1\lambda} \hat{\mathbf{b}}_\lambda - \hat{\delta}\gamma_1$  on  $\hat{\mathfrak{B}}(\rho)$ . Here  $\hat{\gamma}_1 \in C^{l-1}(\hat{\mathfrak{B}}(\rho), \mathbf{F})$  and  $\|\hat{\xi}_1\|_\rho \leq \varepsilon \|\hat{\xi}\|_\rho$ . Now  $\hat{\xi}_1 \in Z^l(\hat{\mathfrak{B}}(\rho), \mathbf{F})$ . The Smoothing Theorem gives  $\hat{\xi}_1 \in Z^l(\hat{\mathfrak{U}}(\rho), \mathbf{F})$  and  $\hat{\eta}_1 \in C^{l-1}(\hat{\mathfrak{B}}(\rho), \mathbf{F})$  such that  $\hat{\xi}_1 = \tilde{\xi}_1 + \hat{\delta}\eta_1$  on  $\hat{\mathfrak{B}}(\rho)$ . Here  $\|\eta_1\|_\rho$  and  $\|\hat{\xi}_1\|_\rho \leq K \|\tilde{\xi}_1\|_\rho < \frac{1}{2} \|\hat{\xi}\|_\rho$ . Now we use  $\hat{\xi}_1$  instead of  $\hat{\xi}_0$  as above and get:  $\hat{\xi}_2 = \hat{\xi}_1 + \hat{\delta}\eta_2 - \sum a_{2\lambda} \hat{\mathbf{b}}_\lambda - \hat{\delta}\gamma_2$ . Here  $\|\hat{\xi}_2\|_\rho$  and  $\|\hat{\eta}_2\|_\rho < \frac{1}{2} \|\hat{\xi}_1\|_\rho < (\frac{1}{2})^2 \|\hat{\xi}\|_\rho$  and  $\|a_{2\lambda}\|_\rho$  and  $\|\gamma_2\|_\rho \leq \frac{K}{2} \|\hat{\xi}\|_\rho$ . Inductively we get:  $\hat{\xi}_n = \hat{\xi}_{n-1} - \sum a_{n\lambda} \hat{\mathbf{b}}_\lambda - \hat{\delta}\gamma_n + \hat{\delta}\eta_n$ . Here  $\|\hat{\xi}_n\|_\rho < 2^{-n} \|\hat{\xi}\|_\rho$ ,  $\|\hat{\eta}_n\|_\rho \leq 2^{-n} \|\hat{\xi}\|_\rho$  and  $\|a_{n\lambda}\|_\rho$  and  $\|\gamma_n\|_\rho \leq 2^{-n+1} \cdot K \|\hat{\xi}\|_\rho$  for  $n = 1, 2, 3, \dots$ . A summation is now possible. We get  $0 = \hat{\xi} - \sum_{n,\lambda} a_{n\lambda} \hat{\mathbf{b}}_\lambda - \sum \hat{\delta}\gamma_n + \sum \hat{\delta}\eta_n$ . We put  $a_\lambda = \sum_n a_{n\lambda}$ ,  $\hat{\eta} = \sum (-\hat{\gamma}_n + \hat{\eta}_n)$  and the theorem follows.

For the proof of the coherence the Main Theorem is needed in a weaker and simpler form.

*Main Theorem (\*)*: There exists a positive  $n$ -tuple  $\rho_2 \leq \rho_0$  and cross-sections  $S_1, \dots, S_r \in \Gamma(E^n(\rho_2), \psi_{(l)}(\mathbf{F}))$  such that any  $S = \psi_{(l)}(\hat{\xi}') \in \Gamma(E^n(\rho'), \psi_{(l)}(\mathbf{F}))$  with  $\hat{\xi}' \in H^l(X(\rho'), \mathbf{F})$  can be written over  $E^n(\rho)$  in the form  $S = \sum_1^n a_\lambda S_\lambda$  with  $a_1, \dots, a_r \in I(E^n(\rho))$ . Here  $\rho \leq \rho_2$  and  $\rho < \rho' \leq \rho_0$ .

*Proof.* Define  $S_\lambda = \psi_{(l)}(\hat{\mathbf{b}}_\lambda | X(\rho_2))$ . The cross-section  $S$  can be written in the form  $S = \psi_{(l)}(\hat{\xi}')$  with  $\hat{\xi}' \in Z^l(\hat{\mathfrak{U}}'(\rho'), \mathbf{F})$ . We put  $\hat{\xi} = \hat{\xi}' | \hat{\mathfrak{U}}(\rho)$ . Then  $\|\hat{\xi}\|_\rho < \infty$  and we have the representation  $\hat{\xi} = \sum a_\lambda \hat{\mathbf{b}}_\lambda + \hat{\delta}\eta$ . For the cohomology classes we get  $\hat{\xi} = \sum a_\lambda \hat{\mathbf{b}}_\lambda$  and for the images  $S | E^n(\rho)$ , this gives  $S | E^n(\rho) = \psi_{(l)}(\hat{\xi}) = \sum a_\lambda S_\lambda$ .

The immediate consequence of this form of the Main Theorem is that the stalk of  $\psi_{(l)}(\mathbf{F})$  at the origin (and hence at every point of course) is finitely generated. However this is not yet the full coherence of  $\psi_{(l)}(\mathbf{F})$ . Nevertheless, the Main Theorem above contains all that is essential, and the rest of the proof is not difficult. We refer to [1, pp. 54-58], or to Knorr [2] for details.