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define a parallelepiped $\Pi^{(p)}$ in E^l which we shall call the p-th *pseudocompound* of the parallelepiped Π defined by (8.4).

Remarks. Mahler (1955) defined the p-th *compound* of any symmetric convex set, and the pseudocompound of a parallelepiped is closely related to its compound. But the compound of a parallelepiped is not necessarily a parallelepiped. Except for the notation, the $(n-1)$-st pseudocompound is the same as the dual of a parallelepiped, and hence the results of the last subsection may be interpreted as special cases of the results of the present subsection.

THEOREM 8D (Mahler 1955). *Let $\lambda_1, \ldots, \lambda_n$ and $\nu_1, \ldots, \nu_l$ be the successive minima of a parallelepiped Π and of its p-th pseudocompound $\Pi^{(p)}$, respectively. For $\sigma \in C(n, p)$ put $\lambda_\sigma = \prod_{i \in \sigma} \lambda_i$ and order the elements of $C(n, p)$ as $\sigma_1, \ldots, \sigma_l$ such that $\lambda_{\sigma 1} \leqq \ldots \leqq \lambda_{\sigma l}$. Then*

$$\nu_j \gg \ll \lambda_{\sigma j} \qquad (j = 1, \ldots, l)\,.$$

Moreover, if $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent integer points with (8.1), *i.e. with $|L_i(\mathbf{x}_j)| \leqq \lambda_j R_i$ $(i, j = 1, \ldots, n)$, and if for $\tau = \{j_1, \ldots, j_p\}$ in $C(n, p)$ we put $\mathbf{X}_\tau = \mathbf{x}_{j1} \wedge \ldots \wedge \mathbf{x}_{jp}$, then*

$$|L_\sigma^{(p)}(\mathbf{X}_\tau)| \ll \lambda_\tau R_\sigma \qquad (\sigma, \tau \in C(n, p))\,.$$

9. OUTLINE OF THE PROOF OF THE THEOREMS ON SIMULTANEOUS APPROXIMATION TO ALGEBRAIC NUMBERS

9.1. Let us see what happens if we try to generalize Roth's proof to prove, say, Corollary 7B. In Roth's proof we constructed a polynomial $P(x_1, \ldots, x_m)$ in m variables $x_1, \ldots, x_m$ which had a zero of high order at $(\alpha, \ldots, \alpha)$. Hence the natural thing to try would be

(a) to construct a polynomial $P(x_{11}, \ldots, x_{1l}; \ldots; x_{m1}, \ldots, x_{ml})$ in ml variables of total degree $\leqq r_h$ in each block of variables $x_{h1}, \ldots, x_{hl}$ $(h = 1, \ldots, m)$ with a zero of high order at $(\alpha_1, \ldots, \alpha_l; \ldots; \alpha_1, \ldots, \alpha_l)$. Then

(b) one would have to show that if each of m given rational l-tuples $\left(\frac{p_{h1}}{q_h}, \ldots, \frac{p_{hl}}{q_h}\right)$ $(h = 1, \ldots, m)$ satisfies (7.2), then P also has a zero of high order at

$$\left(\frac{p_{11}}{q_1}, \ldots, \frac{p_{1l}}{q_1}; \ldots; \frac{p_{m1}}{q_m}, \ldots, \frac{p_{ml}}{q_m}\right).$$

Finally

(c) one would have to show that under suitable conditions P cannot have a high zero at such a rational point.

If we proceed in this fashion, we encounter difficulties in (c). In Roth's Lemma 3C it was essential that P had rather different degrees in its variables and that the denominators in $\frac{p_1}{q_1}, \ldots, \frac{p_m}{q_m}$ increased very fast. In our present situation the first l denominators are equal, so that Roth's Lemma does not apply. The example $m = 1$, $l = 2$, $P(x_1, x_2) = (x_1 - x_2)^r$ shows that we cannot expect to have a lemma similar to Roth's in our present context, since P has a zero of order as high as r at every point (ξ, ξ).

The polynomial P is defined on $E^l \times \ldots \times E^l$ (m copies). While it is difficult to say much about the order of vanishing of P at rational points $\mathbf{r}_1 \times \ldots \times \mathbf{r}_m$, it is easier to show that P cannot have a zero of high order on certain linear manifolds $\mathcal{M}_1 \times \ldots \times \mathcal{M}_m$ where each $\mathcal{M}_h$ is a rational (i.e. defined by a linear equation with rational coefficients) hyperplane in E^l. We can illustrate this when $m = 1$. Namely, $\mathcal{M}_1$ is defined by an equation $a_0 + a_1 x_1 + \ldots + a_l x_l = 0$ which can be normalized such that $a_0, a_1, \ldots, a_l$ are coprime rational integers. If $P(x_1, \ldots, x_l)$ has a zero of order $\geqq i$ on $\mathcal{M}_1$ (i.e. P has a zero of order $\geqq i$ at every point of $\mathcal{M}_1$), then $P(x_1, \ldots, x_l) = (a_0 + a_1 x_1 + \ldots + a_l x_l)^i \, R(x_1, \ldots, x_l)$, where R has integer coefficients by Gauss' Lemma. It follows that

$$(H(M))^i \leqq H(P) \tag{9.1}$$

where $H(M)$ is the height of $M(\mathbf{x}) = a_0 + a_1 x_1 + \ldots + a_l x_l$. This inequality provides a good upper bound for i if $H(M)$ is large.

9.2. It will be more convenient to deal with hyperplanes through the origin in E^{l+1} than with hyperplanes in E^l. Hence we shall put

$$n = l + 1 \tag{9.2}$$

and we shall consider polynomials $P(x_{11}, \ldots, x_{1n}; \ldots; x_{m1}, \ldots, x_{mn})$ which are homogeneous of degree r_h in each block of variables $x_{h1}, \ldots, x_{hn}$ $(h = 1, \ldots, m)$. The manifold $\mathcal{M}_1 \times \ldots \times \mathcal{M}_m$ now becomes a subspace defined by $L_1(x_{11}, \ldots, x_{1n}) = \ldots = L_m(x_{m1}, \ldots, x_{mn}) = 0$, where each L_h is a not identically vanishing linear form in $x_{h1}, \ldots, x_{hn}$ $(h=1, \ldots, m)$. The polynomial P vanishes on $\mathcal{M}_1 \times \ldots \times \mathcal{M}_m$ precisely if it lies in the ideal generated by $L_1, \ldots, L_m$. A suitable definition of the index is now as follows.

Let $L_h = L_h(x_{h1}, \ldots, x_{hn})$ $(h=1, \ldots, m)$ be not identically vanishing linear forms. For positive integers $r_1, \ldots, r_m$ and for $c \geqq 0$ let $\mathcal{T}(c)$ be the ideal generated by the products $L_1^{i_1} \ldots L_m^{i_m}$ with

$$\frac{i_1}{r_1} + \ldots + \frac{i_m}{r_m} \geqq c .$$

The index of P with respect to $(L_1, \ldots, L_m; r_1, \ldots, r_m)$ is the largest value of c such that $P \in \mathcal{T}(c)$ if P is not identically zero, and it is $+\infty$ if P is identically zero.

9.3. Now suppose that $L(\mathbf{x}) = \alpha_1 x_1 + \ldots + \alpha_n x_n$ has real algebraic coefficients. In analogy with Lemma 3A in step (a) in the proof of Roth's Theorem, one can construct a polynomial P as above which is not identically zero and which has not too large rational integer coefficients, such that P has index at least

$$\left(\frac{1}{n} - \varepsilon\right) m ,$$

with respect to $(L, \ldots, L; r_1, \ldots, r_m)$. Here L really occurs with m different meanings; namely, the h-th copy of L means $\alpha_1 x_{h1} + \ldots + \alpha_n x_{hn}$ $(h=1, \ldots, m)$. Perhaps it should be explained why the factor $\frac{1}{2} - \varepsilon$ in Lemma 3A is now replaced by $\frac{1}{n} - \varepsilon$. A form P in mn variables $x_{11}, \ldots, x_{1n}; \ldots; x_{m1}, \ldots, x_{mn}$ is also a form in $L, x_{12}, \ldots, x_{1n}; \ldots; L, x_{m2}, \ldots, x_{mn}$ provided $\alpha_1 \neq 0$ (and where L occurs with different meanings again). Now for " most " monomials in $L, x_{12}, \ldots, x_{1n}; \ldots; L, x_{m2}, \ldots, x_{mn}$ the degree in L will be about $\frac{1}{n}$ times the total degree of the monomial, and hence will be greater than $\left(\frac{1}{n} - \varepsilon\right)$ times the total degree of the monomial.

But a result with only one linear form L is not enough. In general, say when dealing with General Roth Systems, one has n linear forms $L_1, \ldots, L_n$ to start with, and one can deal with them simultaneously. The following result now replaces Lemma 3A.

LEMMA 9A. *Let $L_1, \ldots, L_n$ be not identically vanishing linear forms with real algebraic coefficients. Suppose $\varepsilon > 0$. Then if $m > m_0 (L_1, \ldots, L_n; \varepsilon)$ and if $r_1, \ldots, r_m$ are positive integers, there is a polynomial $P(x_{11}, \ldots, x_{1n}; \ldots; x_{m1}, \ldots, x_{mn}) \neq 0$ with rational integer coefficients such that*

(i) *P is homogeneous in $x_{h1}, \ldots, x_{hn}$ of degree r_h $(h=1, \ldots, m)$.*

(ii) *P has index $\geqq \left(\frac{1}{n} - \varepsilon\right) m$ with respect to $(L_i, \ldots, L_i; r_1, \ldots, r_m)$*
$(i=1, \ldots, n)$.

(iii) *$H(P) \leqq B^{r_1 + \cdots + r_m}$ where $B = B(L_1, \ldots, L_m)$.*

This takes care of generalizing part (a) of Roth's proof. We have chosen our definition of the index such that (c) has a chance of going through, and in fact one can derive from Roth's Lemma 3C a more general lemma that applies in our situation. Namely, if $M_1(\mathbf{x}), \ldots, M_m(\mathbf{x})$ are linear forms with rational integer coefficients, then under suitable conditions the index of P with respect to $(M_1, \ldots, M_m; r_1, \ldots, r_m)$ is $\leqq \varepsilon$.

9.4. If thus remains to deal with part (b). Suppose, say, that we want to derive a criterion for General Roth Systems as defined in §7.3. Suppose $L_1, \ldots, L_n$ are linear forms with real algebraic coefficients and suppose $\gamma_1 + \ldots + \gamma_n = 0$. Suppose there is a $\delta > 0$ and there are arbitrarily large values of Q for which there is an integer point $\mathbf{x} \neq \mathbf{0}$ with $|L_i(\mathbf{x})| < Q^{\gamma_i - \delta}$ $(i=1, \ldots, n)$. Assume in particular that this is true for $Q = Q_1, \ldots, Q_m$ and with integer points $\mathbf{x}_1, \ldots, \mathbf{x}_m$, respectively. An argument like the one used in the proof of Lemma 3B shows that if suitable auxiliary conditions are satisfied, then the polynomial P of Lemma 9A does in fact have

$$P(\mathbf{x}_1, \ldots, \mathbf{x}_m) = 0 .$$

But this is not what we really need. Namely, we need a rational subspace of the type $\mathcal{M}_1 \times \ldots \times \mathcal{M}_m$ where each $\mathcal{M}_h$ is a hyperplane of E^n, such that P vanishes on this subspace.

There is a way out of this difficulty, although it is a rather costly one. Namely, we have to assume that for each Q_h $(h=1, \ldots, m)$ there is not just one but there are

$$l = n - 1$$

linearly independent integer points $\mathbf{x}_h^{(1)}, \ldots, \mathbf{x}_h^{(l)}$ with

(9.3) $\quad | L_i(\mathbf{x}_h^{(j)}) | \leqq Q_h^{\gamma_i - \delta} \quad (i=1, \dots, n; j=1, \dots, l; h=1, \dots, m)$.

Now if $\mathcal{M}_h$ is the hyperplane through $\mathbf{0}$ spanned by $\mathbf{x}_h^{(1)}, \dots, \mathbf{x}_h^{(l)}$ $(h = 1, \dots, m)$, then one can show that P vanishes on $\mathcal{M}_1 \times \dots \times \mathcal{M}_m$. In fact one can show that if M_h is the linear form defining $\mathcal{M}_h$ $(h=1, \dots, m)$, then the index of P with respect to $(M_1, \dots, M_m; r_1, \dots, r_m)$ is $\geqq m\varepsilon$, which in conjunction with (c) gives the desired contradiction.

9.5. But what have we really shown now? The inequalities

(9.4) $$| L_i(\mathbf{x}) | \leqq Q^{\gamma_i} \qquad (i=1, \dots, n)$$

define a parallelepiped. The presence of $l = n-1$ linearly independent integer points $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(l)}$ with $| L_i(\mathbf{x}^{(j)}) | \leqq Q^{\gamma_i - \delta}$ $(i=1, \dots, n; j=1, \dots, l)$ means that the $(n-1)$ st minimum $\lambda_{n-1} = \lambda_{n-1}(Q)$ satisfies $\lambda_{n-1} \leqq Q^{-\delta}$. The inequalities (9.3) mean precisely that $\lambda_{n-1}(Q) \leqq Q^{-\delta}$ for $Q = Q_1, Q_2, \dots, Q_m$. Thus we obtain a theorem about λ_{n-1}:

THEOREM 9B. *(Theorem on the next to last minimum). Suppose* $n \geqq 2$ *and* $L_1, \dots, L_n$ *are linearly independent linear forms with real algebraic coefficients, and suppose* $\overset{*}{L}_1, \dots, \overset{*}{L}_n$ *are their duals. Suppose* $\delta > 0$, *suppose* $\gamma_1 + \dots + \gamma_n = 0$, *and let* Σ *be the set of integers i in* $1 \leqq i \leqq n$ *for which*

$$\gamma_i + \delta \geqq 0 .$$

There is a $Q_0 = Q_0(L_1, \dots, L_n; \gamma_1, \dots, \gamma_n; \delta)$ *with the following property: Let* $\lambda_1 = \lambda_1(Q), \dots, \lambda_n = \lambda_n(Q)$ *be the successive minima of the parallelepiped* $\Pi(Q)$ *given by* (9.4). *Then for* $Q > Q_0$ *either*

(9.5) $$\lambda_{n-1} > Q^{-\delta}$$

or

(9.6) $$\overset{*}{L}_i(\overset{*}{\mathbf{x}}_n) = 0 \textit{ for every } i \in \Sigma ,$$

where $\overset{*}{\mathbf{x}}_1, \dots, \overset{*}{\mathbf{x}}_n$ *are the duals* [1]) *to linearly independent integer points* $\mathbf{x}_1, \dots, \mathbf{x}_n$ *with* $\mathbf{x}_j \in \lambda_j \Pi$ $(j=1, \dots, n)$.

It was clear from the discussion above that some inequality such as (9.5) would result. The hyperplanes $\mathcal{M}$ of the discussion above were spanned by $\mathbf{x}_1, \dots, \mathbf{x}_{n-1}$ (but with the notation $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(l)}$), and hence the coefficients

[1]) I.e. they satisfy $\mathbf{x}_i \mathbf{x}_j^* = \delta_{ij}$ $(i, j=1, \dots, n)$.

in the defining equation for $\mathcal{M}$ are proportional to $\mathbf{x}_n^*$. The alternative (9.6) had to be put in to allow for the possibility that $\mathcal{M}$ behaves in a somewhat degenerate fashion. In most cases, e.g., if the coefficients of some L_i^* with $i \in \Sigma$ are linearly independent over the rationals, then no integer point $\mathbf{x} \neq \mathbf{0}$ can satisfy (9.6), and then (9.5) must hold.

Theorem 9B gives information on λ_{n-1} rather than on λ_1. In what follows, transference theorems will be used to gain information on λ_1.

9.6. Theorem 9B says that if Q is large and $\lambda_{n-1} < Q^{-\delta}$, then $\mathbf{x}_n^*$ must lie in a certain subspace. The inequality (8.7) of Mahler's Theorem 8C further restricts the possibilities for $\mathbf{x}_n^*$. A combination of these results yields

COROLLARY 9C. *Suppose* $L_1, \ldots, L_n, \gamma_1, \ldots, \gamma_n, \delta, \mathbf{x}_1 = \mathbf{x}_1(Q), \ldots, \mathbf{x}_n = \mathbf{x}_n(Q), \mathbf{x}_1^* = \mathbf{x}_1^*(Q), \ldots, \mathbf{x}_n^* = \mathbf{x}_n^*(Q)$ *are as above. Suppose there are arbitrarily large values of* Q *with*

$$\lambda_{n-1} < Q^{-\delta}. \tag{9.7}$$

Then there is a fixed vector $\mathbf{c}$ *and there are arbitrarily large values of* Q *with* (9.7) *and with* $\mathbf{x}_n^*(Q) = \mathbf{c}$.

Next, the condition (9.7) will be replaced by

$$\lambda_{n-1} < Q^{-\delta}\lambda_n. \tag{9.8}$$

The latter condition usually is milder, since $\lambda_n \gg 1$ by (8.5).

THEOREM 9D. *(Theorem on the last two minima). Suppose* $L_1, \ldots, L_n, \gamma_1, \ldots, \gamma_n, \delta, \mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_1^*, \ldots, \mathbf{x}_n^*$ *are as above. Suppose there are arbitrarily large values of* Q *with* (9.8). *Then there are arbitrarily large values of* Q *with* (9.8) *and with* $\mathbf{x}_n^*(Q) = \mathbf{c}$, *where* $\mathbf{c}$ *is a fixed vector.*

To prove this theorem one needs Davenport's Lemma (Theorem 8B). Namely, put $\rho_0 = (\lambda_1 \ldots \lambda_{n-2}\lambda_{n-1}^2)^{1/n}$ and

$$\rho_1 = \rho_0/\lambda_1, \ldots, \rho_{n-1} = \rho_0/\lambda_{n-1}, \text{ but } \rho_n = \rho_0/\lambda_{n-1}.$$

By Davenport's Lemma we can compare the successive minima $\lambda_1, \ldots, \lambda_n$ of Π with the successive minima $\lambda_1', \ldots, \lambda_n'$ of another parallelepiped Π'. We have $\lambda_j' \gg\ll \rho_j\lambda_j$ $(j=1, \ldots, n)$ and $\rho_0 \ll \lambda_1' \ll \ldots \ll \lambda_{n-1}' \ll \rho_0 \ll (\lambda_{n-1}/\lambda_n)^{1/n} \ll Q^{-\delta/n}$ by (8.5) and (9.8). Hence $\lambda_{n-1}' < Q^{-\delta/(2n)}$ if Q is large, and applying Corollary 9C to Π' we see that $\mathbf{x}_n^{*\prime}(Q)$ is the same for arbitrarily large values of Q, which in turn (by the last assertion of Davenport's Lemma) implies that $\mathbf{x}_n^*(Q)$ is the same for certain arbitrarily large values of Q.

9.7. THEOREM 9E. *(Subspace Theorem). Suppose* $L_1, \ldots, L_n, \gamma_1, \ldots, \gamma_n, \delta$, $\mathbf{x}_1(Q), \ldots, \mathbf{x}_n(Q)$ *are as above. Suppose there is a* d *in* $1 \leqq d \leqq n-1$ *such that*

$$\lambda_d < \lambda_{d+1} Q^{-\delta} \tag{9.9}$$

for certain arbitrarily large values of Q. *Then there is a fixed rational subspace* S^d *of dimension* d *such that for some arbitrarily large values of* Q *with* (9.9), *the points*

$$\mathbf{x}_1(Q), \ldots, \mathbf{x}_d(Q) \text{ lie in } S^d .$$

For the proof put $p = n - d$ and construct the linear forms $L_\sigma^{(p)}$ as in §8.4. Also put $\Gamma_\sigma = \sum_{i \in \sigma} \gamma_i$. The inequalities

$$| L_\sigma^{(p)}(\mathbf{X}) | \leqq Q^{\Gamma_\sigma} \qquad (\sigma \in C(n, p))$$

define the p-th pseudocompound $\Pi^{(p)}$ of Π. By Mahler's Theorem 8D the last two minima ν_{l-1}, ν_l of this pseudocompound have

$$\nu_{l-1} \gg \ll \lambda_d \lambda_{d+2} \lambda_{d+3} \cdots \lambda_n, \quad \nu_l \gg \ll \lambda_{d+1} \lambda_{d+2} \lambda_{d+3} \cdots \lambda_n,$$

whence $\nu_{l-1} < \nu_l Q^{-\delta/2}$ for large Q by (9.9). An application of Theorem 9D shows that $\mathbf{X}_l^*$ [1]) is the same for some arbitrarily large values of Q. Some algebra combined with the last assertion of Theorem 8D shows that (because of (9.9)) $\mathbf{X}_l^*$ is proportional to $\mathbf{x}_{d+1}^* \wedge \ldots \wedge \mathbf{x}_n^*$. It follows that the subspace S^* spanned by $\mathbf{x}_{d+1}^*, \ldots, \mathbf{x}_n^*$ is the same for some arbitrarily large values of Q. But for these values of Q the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_d$ lie in the orthogonal complement S^d of S^*.

9.8. We shall illustrate the power of the Subspace Theorem by deducing Theorem 7E. Suppose we have $\delta > 0$, $1 \leqq m < n$, m linearly independent linear forms $L_1, \ldots, L_m$ with real algebraic coefficients, and infinitely many integer solutions $\mathbf{x} \neq \mathbf{0}$ of

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[1]) $\mathbf{X}_l^*$ in E^l is defined in terms of $\Pi^{(p)}(Q)$ just as $\mathbf{x}_n^*$ in E^n was defined in terms of $\Pi(Q)$.

$$| L_i(\mathbf{x}) | \leqq | \mathbf{x} |^{-((n-m)/m)-\delta} \qquad (i=1, \ldots, m)\,.$$

We may assume without loss of generality that $L_1, \ldots, L_m, x_1, \ldots, x_{n-m}$ are linearly independent. Put $L_{m+1}(\mathbf{x}) = x_1, \ldots, L_n(\mathbf{x}) = x_{n-m}$. It is easy to see that there is a $\delta' > 0$ and there are arbitrarily large values of Q for which there are solutions $\mathbf{x} \neq \mathbf{0}$ of

$$| L_i(\mathbf{x}) | \leqq Q^{\gamma_i - \delta'} \qquad (i=1, \ldots, n)$$

where $\gamma_1 = \ldots = \gamma_m = -(n-m)/m$ and $\gamma_{m+1} = \ldots = \gamma_n = 1$. For these values of Q one has $\lambda_1 = \lambda_1(Q) < Q^{-\delta'}$. Since $\lambda_1 \leqq \ldots \leqq \lambda_n$ and $1 \ll \lambda_1 \ldots \lambda_n \ll 1$, there is a d with $1 \leqq d \leqq n-1$ and a $\delta'' > 0$ such that

$$\lambda_d < \lambda_{d+1} Q^{-\delta''} \tag{9.10}$$

for arbitrarily large values of Q. Let S^d be the subspace in the conclusion of Theorem 9E.

Let $\Pi^*(Q)$ be the intersection of $\Pi(Q)$ and S^d; this is a symmetric convex set in S^d. Let $\lambda_1^*, \ldots, \lambda_d^*$ be the successive minima of $\Pi^*(Q)$ with respect to the lattice Λ of integer points in S^d, and let $V^* = V^*(Q)$ be the (d-dimensional) volume of $\Pi^*(Q)$. By applying (8.3) to the lattice Λ we obtain

$$1 \ll \lambda_1^* \ldots \lambda_d^* V^* \ll 1\,, \tag{9.11}$$

where the constants in $\ll$ may depend on S^d. There are arbitrarily large values of Q for which $\mathbf{x}_1(Q), \ldots, \mathbf{x}_d(Q)$ lie in S^d, and for these values we have $\lambda_1 = \lambda_1^*, \ldots, \lambda_d = \lambda_d^*$, whence by (8.5) and (9.10),

$$\begin{aligned} \lambda_1^* \ldots \lambda_d^* &= \lambda_1 \ldots \lambda_d = (\lambda_1 \ldots \lambda_d)^{d/n} (\lambda_1 \ldots \lambda_d)^{(n-d)/n} \\ &< (\lambda_1 \ldots \lambda_d)^{d/n} (\lambda_{d+1} \ldots \lambda_n)^{d/n} Q^{-\delta'' d(n-d)/n} \ll Q^{-\delta'' d(n-d)/n} = Q^{-\eta}, \end{aligned}$$

say. In conjunction with (9.11) this yields $V^* \gg Q^{\eta}$.

Now if $L_1, \ldots, L_m$ have rank r on S^d, then

$$V^* \ll Q^{-(r(n-m)/m)+d-r} = Q^{d-(rn/m)}\,.$$

It follows that $d - (rn/m) \geqq \eta > 0$ and that

$$r < dm/n\,.$$

This cannot happen if (7.6) holds, and hence $L_1, \ldots, L_m$ is a Roth System in this case. Since the case of linearly dependent forms $L_1, \ldots, L_m$ is trivial and since the other half of the theorem was proved in §7.3, Theorem 7E is established.