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SOME ALGEBRAIC CALCULATIONS OF WALL GROUPS FOR Z_2

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The surgery obstructions, which play such an important role in the topology of manifolds, are elements of certain groups L_n defined by C. T. C. Wall [8], [9]. Roughly speaking, if we have a map $\varphi: M^n \to N^n$ of degree 1, satisfying certain additional conditions, and we want to apply surgery on φ to modify it into a homotopy equivalence between the two manifolds, we encounter an obstruction $\theta(\varphi)$, which lies in the group $L_n(\pi_1)$.

Here π_1 denotes the fundamental group of N.

A purely algebraic description of the odd dimensional Wall groups can be given as follows.

Let $\wedge$ be an associative ring with unit, provided with an involution (conjugation) —, i.e. with an anti-automorphism of order 2; let further k be a fixed integer. For any positive integer r, let $SU_r = SU_r(\wedge, -, k)$ be the group of $(2r \times 2r)$ — matrices over $\wedge$ of the form

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

($(\alpha, \beta, \gamma, \delta)$ are $(r \times r)$)-matrices), satisfying

(i) $$A \varepsilon A^* = \varepsilon, \varepsilon = \begin{pmatrix} 0 & I \\ (-1)^k I & 0 \end{pmatrix}, \text{ where}$$

I is the identity,

and

(ii) $$\alpha\beta^* = \varphi - (-1)^k \varphi^*, \quad \gamma\delta^* = \psi - (-1)^k \psi^*$$

for some $(r \times r)$ — matrices φ and ψ. Conditions (i), (ii) are equivalent to (i′) and (ii), where

(i′) $$\alpha\delta^* + (-1)^k \beta\gamma^* = I.$$

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We have used above the notation A^* for the conjugate of the transposed A^T of A.

Let the subgroup $RU_r \subset SU_r$ be generated by the matrices in SU_r of the form

$$(0.1) \qquad \sigma_i = \begin{array}{c} \\ \\ i \\ \\ \\ i \\ \\ \end{array} \begin{pmatrix} & i & & & i & \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & (-1)^k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}$$

and

$$(0.2) \quad \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \qquad (0.3) \quad \begin{pmatrix} I & P \\ 0 & I \end{pmatrix} \qquad (0.4) \quad \begin{pmatrix} I & 0 \\ P & I \end{pmatrix}$$

(clearly $UV^* = I$, $P = \varphi - (-1)^k \varphi^*$).

For any $A \in SU_r$ we shall denote by $A \oplus I_{2p}$ the $((r+p) \times (r+p))$ — matrix

$$(0.5) \qquad A \oplus I_{2p} = \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & I & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

The correspondence $A \to A \oplus I_{2p}$ imbeds SU_r into SU_{r+p} and RU_r into RU_{r+p} so that we can form the stable groups

$$(0.6) \qquad RU = \bigcup_r RU_r, \quad SU = \bigcup_r SU_r$$

Then, as shown by Wall, RU is normal in SU, and the quotient

$$(0.7) \qquad L_{2k+1}(\wedge, -) = SU/RU$$

is abelian.

In particular, let $\wedge = Z(\pi)$ be the integral group ring of π, where π is a group, and let $Z^* = \{-1, 1\}$ be the multiplicative group of units of Z. Suppose that $w: \pi \to Z^*$ is a homomorphism. If π is the fundamental group of a manifold M^{2k+1}, then w is defined by $w(g) = -1$ if and only g reverses orientations. Define an involution $\bar{\ }$ in $Z(\pi)$ as the linear extension of $\bar{g} = w(g)\, g^{-1}$, $g \in \pi$. We shall write in this case $L^h_{2k+1}(\pi, w)$ instead of $L_{2k+1}(\wedge, -)$. The surgery obstructions for modifying maps of manifolds with fundamental group π into homotopy equivalences belong to L^h_{2k+1}.

If we are interested in simple homotopy equivalences, the definition has to be somewhat modified, and we obtain a new group $L^s_{2k+1}(\pi, w)$. It goes without saying that for groups π for which the Whitehead group vanishes, the two definitions coincide, and we shall sometimes omit the superscripts. This will be always the case in this paper ($\pi=1$ or $\pi=Z_2$).

It has been known for a long time from topological considerations, that for the trivial group $\pi = 1$, $L_{2k+1}(1) = 0$ [3], [7]. If $\pi = Z_2$, w is either trivial (in this case we shall write $L^+_{2k+1}(Z_2)$ instead of $L_{2k+1}(Z_2, w)$) or w is an isomorphism (notation: $L^-_{2k+1}(Z_2)$). Wall and López de Medrano have shown (mainly by topological methods) that $L^-_{2k+1}(Z_2) = 0$ and $L^+_{2k+1}(Z_2) = 0$ if k is even, whereas for k odd the latter group is Z_2 (see [5], [8] and [10]). Our aim is to recover all these results in a purely algebraic way, by taking as our starting point the above definition of L_{2k+1}. Such computations for the case $\pi = Z_p$, p odd, have been performed by R. Lee [4]. Our treatment is similar, but much more elementary. It has also points of contact with [1], again being on a much more elementary level, due to the simplicity of the group Z_2.

Section 1 is concerned with the easiest case, that of the " non-orientable " group $L^-_{2k+1}(Z_2)$. The key Lemmas 1.5 and 1.6, which are mild generalizations of the euclidean algorithm for integers, are our main tool throughout the whole paper. Another characteristic feature of our treatment, which has been used before [8], is to always view $Z(Z_2)$ as the subring of the direct sum $Z \oplus Z$ of two copies of the ring of integers, consisting of pairs (a_1, a_2) such that $a_1 \equiv a_2$ (2).

In section 2 we continue in the same spirit and prove that $L_{2k+1}(\wedge) = 0$, for $\wedge = Z$, and that $L^+_{2k+1}(Z_2) = 0$ if k is even.

Section 3 contains the proof of the fact that for k odd, $L^+_{2k+1}(Z_2)$ has at most two elements, by pointing out a possible generator of this group, and by proving that it is at most of order 2. The computation is concluded in Section 4, where we define an " Arf invariant " map c of SU onto Z_2. The main difficulty consists in showing that c is a homomorphism; then it turns out that it vanishes on RU. This is achieved by computing mod 8 and by proving some peculiar Lemmas (4.2-4.5) about determinants over Z_8.

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1. When speaking about the ring of integers as a ring with involution, we shall always assume that this involution is the identity. Let Γ and Σ be two rings with involution which as rings are isomorphic to $Z(Z_2)$ and

are defined in the following way. Let R be the subring of $Z \oplus Z$ consisting of all pairs (a, b) such that $a \equiv b\,(2)$. The correspondence $a + b\xi \rightarrow$ $\rightarrow (a+b, a-b)$, where ξ is the generator of Z_2, clearly induces an isomorphism between $Z(Z_2)$ and R. Then Γ is R with the identity involution and Σ is R with the involution $\overline{(a, b)} = (b, a)$. For every ring $\wedge$, let $GL_r(\wedge)$ be the group of invertible matrices of rank r over $\wedge$; if $\mathfrak{A}$ is a two-sided ideal of $\wedge$, $GL_r(\wedge, \mathfrak{A})$ denotes the normal subgroup of $GL_r(\wedge)$ consisting of all matrices $A \equiv I\,(\mathfrak{A})$.

Any matrix over Σ can be written as a pair (A_1, A_2) where the components add and multiply separately. The projection $(A_1, A_2) \rightarrow A_1$ induces a homomorphism

$$q\colon SU_r(\Sigma) \rightarrow GL_{2r}(Z) \tag{1.1}$$

LEMMA 1.1. *The map q is a monomorphism. Its image is $p^{-1}(SU_r(Z_2))$ where*

$$p\colon GL_{2r}(Z) \rightarrow GL_{2r}(Z_2)$$

is the canonical map.

Proof. Noticing that $(A_1, A_2)^* = (A_2^T, A_1^T)$ and interpreting conditions (i) and (ii) in this case, it follows that $(A_1, A_2) \in SU_r(\Sigma)$ if and only if

$$A_1\, \varepsilon\, A_2^T = A_2\, \varepsilon\, A_1^T = \varepsilon, \quad A_1 \equiv A_2\,(2) \tag{1.2}$$

and

$$\alpha_1 \beta_2^T = \varphi_1 - (-1)^k \varphi_2^T, \alpha_2 \beta_1^T = \varphi_2 - (-1)^k \varphi_1^T,$$

$$\gamma_1 \delta_2^T = \psi_1 - (-1)^k \psi_2^T, \gamma_2 \delta_1^T = \psi_2 - (-1)^k \psi_1^T, \tag{1.3}$$

for some matrices φ_1, ψ_1, such that $\varphi_1 \equiv \varphi_2, \psi_1 \equiv \psi_2\,(2)$ where $A_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$.

From (1.2) we obtain $A_2 = \varepsilon\,(A_1^T)^{-1} \varepsilon^{-1}$, which shows that q is injective. Moreover, since pq commutes with the involutions on Σ and on Z_2, it induces a map of $SU_r(\Sigma)$ into $SU_r(Z_2)$, i.e.

$$p(A_1) \in SU_r(Z_2). \tag{1.4}$$

Let now $p(A_1) \in SU_r(Z_2)$. Then $A_2 = \varepsilon\,(A_1^T)^{-T}\,\varepsilon^{-1}$ satisfies $A_1\,\varepsilon\,A_2^T =$ $= \varepsilon, A_1 \equiv A_2(2)$. Moreover one can check that $\alpha_i \beta_j^T, \gamma_i \delta_j^T$ are even on the diagonal, so that we can take e.g. $\varphi_1 = (-1)^{k-1}\varphi_2 = (f_{ij}), f_{ij} = (\alpha_1\beta_2)_{ij}$ if $i < j$, $f_{ii} = \frac{1}{2}(\alpha_1\beta_2)_{ii}$, $f_{ij} = 0$ if $i > j$ which means that $(A_1, A_2) \in SU(\Sigma)$.

We shall from now on write $\widetilde{SU}_r = q(SU_r(\Sigma))$, $\widetilde{RU}_r = q(RU_r(\Sigma))$. The latter group contains all tbe matrices σ_i from (0.1) and also all the matrices

$$\begin{pmatrix} I & P \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ P & I \end{pmatrix}, \quad P \equiv \varphi + \varphi^T (2) \tag{1.5}$$

and the invertible matrices

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}, \quad UV^T \equiv I(2) \tag{1.6}$$

LEMMA 1.2. $RU_r(Z_2) = SU_r(Z_2)$

Proof. Let

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU_r(Z_2).$$

By applying first some permutations of type σ_i, we can assume that $|\alpha| \neq 0$. Then

$$A = RED, R, E, D \in RU_r(Z_2) \tag{1.7}$$

where

$$R = \begin{pmatrix} I & 0 \\ \gamma\alpha^{-1} & I \end{pmatrix}, \quad E = \begin{pmatrix} I & \beta\alpha^T \\ 0 & I \end{pmatrix}, \quad D = \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^T)^{-1} \end{pmatrix}$$

LEMMA 1.3. *The reduction mod 2 map p satisfies*

$$p(\widetilde{RU}_r) = RU_r(Z_2) = SU_r(Z_2).$$

Proof. It is enough to check that the elements of $RU_r(Z_2)$ of form (0.2) belong to the image. This follows however from the fact that over the field Z_2 any non-singular matrix is a product of elementary matrices, which clearly belong to the image.

If $\mathfrak{A}$ is a two-sided ideal of $\wedge$, the group $GL_n(\wedge, \mathfrak{A})$ consists of those invertible matrices A for which $A \equiv I(\mathfrak{A})$; if $\mathfrak{A} = \wedge$, $GL_r(\wedge, \mathfrak{A}) = GL_r(\wedge)$. In particular, Lemma 1.1 implies that

$$GL_{2r}(Z, 2Z) \subset \widetilde{SU}_r \tag{1.9}$$

Clearly $GL_{2r}(Z, 2Z) = \operatorname{Ker} p$ and by Lemma 1.3, this means that

LEMMA 1.4. $\widetilde{SU}_r = GL_{2r}(Z, 2Z) \cdot \widetilde{RU}_r$.

We shall prove now a very elementary result which plays a major role in this paper. Let $a_i, b_i \in Z$; $m, n > 0$; we shall say that $(a_1, b_1)_{(m,n)} \sim (a_2, b_2)$ or that they are (m, n) − equivalent, if (a_2, b_2) can be obtained from (a_1, b_1) by a sequence of operations which consist in replacing (x, y) by $(x, y \pm nx)$ or by $(x \pm my, y)$.

LEMMA 1.5. *Let (a, b) be such that $|a| \neq 0$ is minimal in the corresponding (m, n)-equivalence class. Then*

i) *either na divides b or, for every integer k, we have*

$$|a| \leqq \frac{m}{2} |b + kna|. \tag{1.10}$$

ii) *Moreover, if $mn \leq 4$ and $(a, b) \equiv (1, 0) \bmod 2$, we have $(a, b) \underset{(m,n)}{\sim} (a', 0)$.*

Proof. i) Suppose that, for some k, we have

$$|a| > \frac{m}{2} |b'| \tag{1.11}$$

where $b' = b + kna$. Clearly, $(a, b) \underset{(m,n)}{\sim} (a, b')$. If $na \nmid b$, $b' \neq 0$ and (1.11) implies that for a suitable choice of the sign, $|a'| = |a \pm mb'| < |a|$, which contradicts minimality.

ii) If $na \mid b$, then clearly $(a, b) \sim (a, 0)$. If however $na \nmid b$, by (1.10) we may assume that $|a| \leq \frac{m}{2} |b|$. Since in our case $mb \nmid a$ we can have also $|b| \leq \frac{n}{2} |a|$, which is clearly impossible if $mn \leq 4$ and $|a| \neq |b|$.

LEMMA 1.6. *Let $A \equiv I(2)$ be a $(2r \times 2r)$-matrix. Then by right and left multiplication by elements of $\widetilde{RU}_r \cap GL_{2r}(Z, 2Z)$, A can be diagonalized.*

Proof. Multiplication of a row of A or of a column of A by an even number, followed by its addition to another row or column, keeps A in the same double coset with respect to both $\widetilde{RU}_r$ and $GL_{2r}(Z, 2Z)$. By using repeatedly Lemma 1.5 ii), with $m = n = 2$ and induction on the order of A we can reduce A to a diagonal form.

THEOREM 1.7. $L^{-}_{2k+1}(Z_2) = 0$.

Proof. The identification of the underlying ring R of Σ with $Z(Z_2)$, described at the beginning of this section, carries over the involution of Σ into the involution $\overline{a + b\xi} = a - b\xi$ where ξ is the generator of Z_2. This is exactly the involution for $L^{-}_{2k+1}(Z_2)$. Therefore $L^{-}_{2k+1}(Z_2) = \widetilde{SU}/\widetilde{RU} = SU(\Sigma)/RU(\Sigma)$. According to Lemma 1.4 it is enough to show that $GL_{2r}(Z, 2Z) \subset \widetilde{RU}_r$. This is however an immediate consequence of Lemma 1.6, if we take in it $A \in GL_{2r}(Z, 2Z)$.

2. Let $\mathfrak{A} \subset Z$ be either the unit ideal, or $\mathfrak{A} = 2Z$. In both cases we shall write $GL(Z, \mathfrak{A})$. Lemma 1.5 and 1.6 and, for $\mathfrak{A} = Z$, a classical consequence of the euclidean algorithm can be stated together as

LEMMA 2.1. *Let $q = 1$ if $\mathfrak{A} = Z$ and $q = 2$ if $\mathfrak{A} = 2Z$. Any pair $(a, b) \equiv (1, 0)(\mathfrak{A})$ is (q, q)-equivalent to a pair $(a', 0)$. Moreover, if $\alpha \equiv I(\mathfrak{A})$ then there exist $T_1, T_2 \in GL_n(Z, \mathfrak{A})$ such that $T_1 \alpha T_2$ is diagonal.*

For a fixed k, and for the trivial involution on Z, define the groups $SU_r = SU_r(Z)$ and $RU_r = RU_r(Z)$. RU contains the matrices of the form

$$(2.1) \qquad \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}, \quad UV^T = I, \quad U \in GL_r(Z)$$

$$(2.2) \qquad \begin{pmatrix} I & P \\ 0 & I \end{pmatrix} \text{ and } \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \quad \text{where } P = \varphi - (-1)^k \varphi^T .$$

More general, let $SU_r(Z, \mathfrak{A}) \subset SU_r \cap GL_{2r}(Z, \mathfrak{A})$ consist of matrices A such that

(2.3) $\qquad \alpha\beta^T = \varphi - (-1)^k \varphi^T, \gamma\delta^T = \psi - (-1)^k \psi^T, \varphi \equiv \psi \equiv 0\,(\mathfrak{A})$

Recall that Γ is the same ring as Σ but endowed with the trivial involution. Then $SU_r(\Gamma)$ consists of pairs (A_1, A_2), $A_i \in SU_r$ such that $A_1 \equiv \equiv A_2$, $\alpha_i\beta_i^T = \varphi_i - (-1)^k \varphi_i^T, \gamma_i\delta_i^T = \psi_i - (-1)^k \psi_i^T$ and $\varphi_1 \equiv \varphi_2, \psi_1 \equiv \equiv \psi_2\,(2)$. Then $SU_r(Z, 2Z)$ is isomorphic to the subgroup of $SU_r(\Gamma)$ which consists of pairs (A, I). This shows, by the way, that $SU_r(Z, 2Z)$ is indeed a group.

Let $RU_r(Z, \mathfrak{A})$ be the normal subgroup of $RU_r(Z)$ generated by the matrices (2.1) and (2.2) lying in $SU_r(Z, \mathfrak{A})$ and by their conjugates. It can easily be checked that $RU_r(Z, \mathfrak{A}) \subset SU_r(Z, \mathfrak{A})$.

Lemma 2.2. *$SU_r(Z, \mathfrak{A})$ is generated by $RU_r(Z, \mathfrak{A})$ and by $SU_1(Z, \mathfrak{A})$.*

Proof. We shall assume that $r \geq 2$ and prove that any $A \in SU_r(Z, \mathfrak{A})$ is equivalent mod $RU_r(Z, \mathfrak{A})$ to $A' \oplus I$ where $A \in SU_{r-1}(Z, \mathfrak{A})$. It is enough to show that we can make $a_{rr} = \pm 1$. Indeed, let ε_{ij} denote the matrix with only one non-zero entry $= 1$ at the intersection of the i-th row and j-th column, and let $q \in \mathfrak{A}$. The matrices

(2.3) $$B_i(q) = \begin{pmatrix} I & q\,(\varepsilon_{ri} + (-1)^{k-1}\,\varepsilon_{ir}) \\ 0 & I \end{pmatrix}$$

and their transposed belong to $RU_r(Z, \mathfrak{A})$. Moreover, if $(\bar{a}; \bar{b}) = = (a_{r1}, \ldots a_{rr}; b_{r1}, \ldots, b_{rr})$ then, for $i \neq r$,

(2.4) $$(\bar{a}; \bar{b})\,B_i(q) = (\bar{a}; b_{r1}, \ldots, b_{ri} \pm q a_{rr}, \ldots, b_{rr} + q a_{ri})\,.$$

If $a_{rr} = 1$, then $(\bar{a}; \bar{b})\,B_1(\mp b_{r1}) \ldots B_{r-1}(\mp b_{r,r-1}) = (\bar{a}; 0, \ldots, b'_{rr})$ and one can immediately see that $b'_{rr} = 0$ if k is even and that $b'_{rr} \in 2\mathfrak{A}$ if k is odd, so that

$$(\bar{a}; 0, \ldots, 0, b'_{rr})\,B_r(+b'_{rr}/2) = (\bar{a}; 0)\,.$$

Obviously, there exists $U \in GL_r(Z, \mathfrak{A})$, such that

$$\bar{a}\,U = (0, \ldots, 0, 1)\,.$$

Then, if $UV^T = I$,

$$(\bar{a}; 0)\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = (0, \ldots, 0, 1; 0, \ldots, 0).$$

By left multiplication by another matrix of the same type, we can bring α to the form $\begin{pmatrix} \alpha' & 0 \\ 0 & 1 \end{pmatrix}$. $\alpha\beta^T = \pm \beta\alpha^T$ implies then that $\beta = \begin{pmatrix} \beta' & 0 \\ 0 & 0 \end{pmatrix}$. Similarly, by row operations we can make $\gamma = \begin{pmatrix} \gamma' & 0 \\ 0 & 0 \end{pmatrix}$. Now, the definition of SU_r implies that $\delta = \begin{pmatrix} \delta' & 0 \\ 0 & 1 \end{pmatrix}$.

It remains for us to show that we can make $a_{rr} = \pm 1$. Suppose that the absolute value of a_{rr} is minimal for the equivalence class of A (and positive). It follows from Lemma 2.1 that a_{rr} divides all a_{rj}, since we are allowed to multiply A by any matrix of type $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ where U is arbitrary in $GL_r(Z, \mathfrak{A})$. The same Lemma also implies that $a_{rr} \mid b_{ri}, i < r$. Indeed, in addition to (2.4) notice that

$$(2.5) \qquad (\bar{a}; \bar{b})\, B_i(q)^T = (a_{r1}, \ldots, a_{ri} \pm qb_{rr}, \ldots, a_{rr} + qb_{ri}; \bar{b}),$$

so that any (q, q)-equivalence for the pair (a_{rr}, b_{ri}) can be realized. To conclude the proof, we may, after reducing first $\bar{a}$ to $(0, \ldots, 0, a_{rr})$ as above, obtain $\bar{b} = (0, \ldots, b_{rr})$. Then if k is even $b_{rr} = 0$, if k is odd

$$(\bar{a}; \bar{b})\, B_{r-1}(q) = (0, \ldots + qb_{rr}, a_{rr}; \bar{b})$$

where a_{rr} and qb_{rr} must be relatively prime (if we take $q = 1$ if $\mathfrak{A} = Z$ or $q = 2$ if $\mathfrak{A} = 2Z$). By virtue of the preceding divisibility remarks, this can happen only if $a_{rr} = \pm 1$.

LEMMA 2.3. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Then, except for the case when k is odd and* $\mathfrak{A} = 2Z$, $A \in RU_1(Z, \mathfrak{A})$.

Proof. a) Let k be even. Then $ab = -ba$ and either $a = 0$ or $b = 0$. Since we may always assume that a is odd, $b = 0$, and $A \in RU_1$.

b) Let k be odd and $A \in RU_1(Z)$. Then by multiplication with σ_1 we can assume that $A \in GL_r(Z, 2Z)$. The argument of Lemma 1.6 shows then that $A \in RU_1 = RU_1(Z)$.

THEOREM 2.4. $L^h_{2k+1}(1) = 0$.

Proof. Follows from Lemmas 2.2 and 2.3, since $Z(\pi) = Z$ if $\pi = 1$.

The group $RU_r(\Gamma) \subset SU_r(\Gamma)$ is generated by pairs (σ_i, σ_i) where σ_i is defined in (0.1), and (A_1, A_2), $A_1 \equiv A_2 \bmod 2$, of the type

$$A_i = \begin{pmatrix} U_i & 0 \\ 0 & V_i \end{pmatrix}, \quad U_i V_i^T = I,\ U_i, V_i \in GL_r(Z)\,; \tag{2.7}$$

$$A_i = \begin{pmatrix} I & P_i \\ 0 & I \end{pmatrix} \ \textit{or} \ A_i = \begin{pmatrix} I & 0 \\ P_i & I \end{pmatrix}, P_i = \varphi_i - (-1)^k \varphi_i^T, \varphi_1 \equiv \equiv \varphi_2\,(2). \tag{2.8}$$

We shall need

LEMMA 2.5. *Let $A = (A_1, A_2)$, $B = (B_1, B_2) \in SU_r(\Gamma)$. Then A and B belong to the same double coset with respect to $RU_r(\Gamma)$ if and only if there exist matrices $S, T \in RU_r(Z, 2Z)$, $W \in RU_r(Z)$ such that*

$$SWA_1A_2^{-1}W^{-1}T = B_1B_2^{-1}\,. \tag{2.9}$$

Proof. Let (2.9) be satisfied. By definition of $RU_r(Z, 2Z)$

$$S = S_1 s_1 S_1^{-1} \dots S_m s_m S_m^{-1},\ T = T_1 t_1 T_1^{-1} \dots T_m t_m T_m^{-1}$$

where $S_i, T_i \in RU_r(Z)$ and $s_i, t_i \in SU_r(Z, 2Z)$ are generators of the type (2.1)-(2.2).

It is enough to assume $n = m = 1$. By Theorem 2.4, $A_2, B_2 \in RU_r(Z)$, so that (A_2, A_2), $(B_2, B_2) \in RU_r(\Gamma)$; since (s_i, I), $(t_i, I) \in RU_r(\Gamma)$, we have, assuming (2.9),

$$\begin{aligned}(B_1, B_2) = {} & (S_1, S_1)(s_1, I)(S_1^{-1}, S_1^{-1})(W, W)(A_1, A_2)(A_2^{-1}, A_2^{-1}) \\ & (W^{-1}, W^{-1})(T_1, T_1)(t_1, I)(T_1^{-1}, T_1^{-1})(B_2, B_2)\end{aligned}$$

where all the factors besides (A_1, A_2) are in $RU_r(\Gamma)$. Now let

$$(B_1, B_2) = (R_1, R_2)(A_1, A_2)(Q_1, Q_2) \tag{2.10}$$

where (R_1, R_2), $(Q_1, Q_2) \in RU_r(\Gamma)$. Then, as above, it is enough to assume that (R_1, R_2), (Q_1, Q_2) are generators of one of the types (2.7), (2.8) or (σ_i, σ_i). It is easy then to see that

$$R_1R_2^{-1}, Q_1Q_2^{-1} \in RU(Z, 2Z), A_2 \in RU_r(Z), R_1 \in RU_r(Z);$$

$$B_1B_2^{-1} = (R_1A_1A_2^{-1}R_1^{-1})(R_1A_2Q_1Q_2^{-1}A_2^{-1}R_1^{-1})(R_1R_2^{-1})$$

which means that (2.9) holds.

THEOREM 2.6. *If k is even,* $L_{2k+1}^{+}(Z_2) = 0$.

Proof. Let $(A_1, A_2) \in SU_r(\Gamma)$. Then $A_1A_2^{-1} \in SU_r(Z, 2Z)$. But Lemmas 2.2, 2.3 imply that $SU_r(Z, 2Z) = RU_r(Z, 2Z)$; by the previous Lemma, this means that $(A_1, A_2) \in RU_r(\Gamma)$. Therefore $L_{2k+1}(\Gamma) = 0$ (recall that Γ is isomorphic to $Z(Z_2)$ with the trivial involution).

3. From now on k will be assumed odd. Let $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Then

$$T^4, R^4 \in RU_1(Z, 2Z), \quad T^2, R^2 \in RU_1(Z). \tag{3.1}$$

LEMMA 3.1. *$SU_1(Z, 2Z)$ is generated by the matrices $-I$, T^4 and*

$$T^iR^4T^{-i}, \quad i = 0, 1, 2, 3$$

Proof. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU_1(Z, 2Z)$ if and only if $|A| = 1$ and $b \equiv$ $\equiv c \equiv 0 \mod 4$,

$$TAT^{-1} = \begin{pmatrix} a-b & b \\ c+a-b-d & d+b \end{pmatrix} \qquad AR^4 = \begin{pmatrix} a & b+4a \\ c & d+4c \end{pmatrix}$$

This means that by conjugation by powers of T and by multiplication by powers of R, we can transform (a, b) into any $(1, 4)$ equivalent pair. This implies by Lemma 1.5 ii, that by our operations we can bring A to the form B, where $B = T^{4i}$ or $B = (-I)\,T^{4i}$. Conversely A can be obtained from B by the same type of operations. The result follows by noticing that

$$T^{4k+i}AT^{-4k-i} = (T^{4k})(T^iAT^{-i})\,T^{-4k}, \quad \textit{where } i = 0, 1, 2, 3,$$

LEMMA 3.2. *Any element $(A_1, A_2) \in SU_1(\Gamma)$ is equivalent, mod $RU(\Gamma)$ to $D^m = (D_1^m, I)$ where*

(3.2) $$D_1 = TR^4T^{-1} = \begin{pmatrix} -3 & 4 \\ -4 & 5 \end{pmatrix}$$

Proof. By Lemma 2.5, we see that $(A_1, A_2) \sim (A, I)$ where $A = A_1A_2^{-1}$; by the same Lemma and the Lemmas 2.2 and 3.1, we may assume that $A = T^iR^4T^{-i}$, $i \leq 3$. But in view of (3.1), Lemma 2.5 implies that $(T^2R^4T^{-2}, I) \in RU(\Gamma)$ and that

$$(T^3R^4T^{-3}, I) \sim (D_1, I)$$

PROPOSITION 3.3. *The group $L_{2k+1}(\Gamma)$ for k odd has at most 2 elements.*

Proof. It is enough to show that $D^2 \in RU(\Gamma)$. Let $E_1 = \begin{pmatrix} 13 & 4 \\ 16 & 5 \end{pmatrix} = D_1R^4$. Then $(E_1, I) \sim D$ and therefore $(A_1, I) \sim D^2$, where

(3.3) $$A_1 = \begin{pmatrix} 13 & 0 & 4 & 0 \\ 0 & -3 & 0 & 4 \\ 16 & 0 & 5 & 0 \\ 0 & -4 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & -4 & 0 & 5 \end{pmatrix} \begin{pmatrix} 13 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 16 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now let $U = \begin{pmatrix} 1 & 2 \\ 6 & 13 \end{pmatrix} \in GL_2(Z, 2Z)$. Then $B = \begin{pmatrix} U & 0 \\ 0 & (U^T)^{-1} \end{pmatrix} \in$

$\in RU_2(Z, 2Z)$

and

$$BA_1B^T = \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & -39 & 0 & 4 \\ & M & & N \end{pmatrix},$$ where M, N are of no interest to us.

Now set

$$C = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \in RU_r(Z),$$ where $P = \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix}$, and

$$CBA_1B^TC^{-1} = \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \\ & M' & & N' \end{pmatrix}$$

is easily seen to belong to $RU(Z, 2Z)$, and by Lemma 2.5 this implies that $D^2 \sim (I, 1)$.

4. We shall now define a map $c: SU(\Gamma) \to Z_2$ which will ultimately turn out to be a homomorphism which vanishes on $RU(\Gamma)$, and we shall show that $c(D) = I$, where D is the generator defined in the previous section.

Let $(A_1, A_2) \in SU_r(\Gamma)$. Then $A_1 A_2^{-1} \in SU_r(Z, 2Z)$. In general, if $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU_r(Z, 2Z)$, we define $\kappa(A)$ as the determinant $|\alpha|$ taken mod 8. $\kappa(A)$ is a unit in Z_8, and we set $c(A_1, A_2) = \kappa(A_1 A_2^{-1})$ modulo the trivial units ± 1. Thus c takes values in the quotient $Z_8^*/Z^* \approx Z_2$. Clearly c is well defined on the stable group $SU(\Gamma)$.

Since we are calculating mod 8, it is convenient to consider the groups $SU_r(Z_8)$, $RU_r(Z_8)$, $SU_r(Z_8, Z_4)$, $RU_r(Z_8, Z_4)$, defined analogously to $SU_r(Z)$, $RU_r(Z)$, $SU_r(Z, 2Z)$, $RU_r(Z, 2Z)$.

LEMMA 4.1. i) $SU_r(Z_8, Z_4) = RU_r(Z_8, Z_4)$.

ii) $SU_r(Z_8) = RU_r(Z_8)$.

Proof. i) Let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU_r(Z_8, Z_4)$. Then $|\alpha| \equiv 1 \bmod 2$ and is a unit in Z_8, so that

$$A = REF, \; R, E, F \in RU_r(Z_8, Z_4) \tag{4.1}$$

where $R = \begin{pmatrix} I & 0 \\ \gamma\alpha^{-1} & I \end{pmatrix}$, $E = \begin{pmatrix} I & \beta\alpha^T \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^T)^{-1} \end{pmatrix}$.

ii) Similar to i), except that we first have to apply some permutations σ_i to make $|\alpha|$ odd.

LEMMA 4.2. *Let $A = I + 2B$ be a matrix over Z_8 and let $Sp\, B$ denote the trace of B and B_{ij}, $i < j$, be the minor*

$$\begin{vmatrix} b_{ii} & b_{ij} \\ b_{ji} & b_{jj} \end{vmatrix}$$

of B. Then $|A| = 1 + 2\, Sp\, B + 4 \sum_{i<j} B_{ij}$

Proof. Left to the imagination of the reader.

LEMMA 4.3. *If* $A', A'' \in SU_r(Z_8, Z_4)$, *then* $\kappa(A'A'') = \kappa(A')\kappa(A'')$.

Proof. Let

$$A' = \begin{pmatrix} I + 2M' & 2N' \\ 2R' & I + 2T' \end{pmatrix}, \quad A'' = \begin{pmatrix} I + 2M'' & 2N'' \\ 2R'' & I + 2T'' \end{pmatrix}$$

$A = A'A''$, $\alpha = (I+2M')(I+2M'') + 4N'R''$. By Lemma 4.2, $\kappa(A) =$ $= |I + 2M'|\,|I + 2M''| + 4\,Sp(N'R'') = n(A')\kappa(A'') + 4\,Sp(N'R'')$.

It is therefore enough to show that $Sp(N'R'') \equiv 0 \bmod 2$. By definition of $SU_r(Z_8, Z_4)$ we have

$$(I+2M')\,2N'^T = 2(\varphi + \varphi^T)$$

and therefore

$$(I+2M')\,N'^T \equiv N'^T \equiv \varphi + \varphi^T \equiv N' \; mod\; 2\,.$$

Similarly,

$$R'' \equiv \psi + \psi^T \; mod\; 2\,.$$

Then, $Sp(N'R'') \equiv Sp(\varphi\psi + \varphi\psi^T + \varphi^T\psi + \varphi^T\psi^T) \equiv 0 \bmod 2$, since for any P, Q, $Sp(PQ) = Sp(QP) = Sp(P^TQ^T)$.

LEMMA 4.4. *If* $A \in SU(Z_8)$, $S \in SU(Z_8, Z_4)$, *then* $\kappa(ASA^{-1}) =$ $= \pm\,\kappa(S)$.

Proof. It is easy to check that $ASA^{-1} \in SU_r(Z_8, Z_4)$, so that, by Lemmas 4.3 and 4.1, it is enough to verify the assertion for generators of $RU_r(Z_8)$ and $RU_r(Z_8, Z_4)$. But any generator $\begin{pmatrix} I & P \\ O & I \end{pmatrix}$ of $RU_r(Z_8)$ is a product of elementary generators of the same form, with $P = \varepsilon_{ij} + \varepsilon_{ji}$ where ε_{ij} has only one non-zero entry 1 at the intersection of the i-th row and the j-th column. In the following list of all possible combinations, $\kappa(ASA^{-1})$ is the determinant of the matrix R_i:

$A \diagdown S$	$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$	$\begin{pmatrix} I & P \\ 0 & I \end{pmatrix}$	$\begin{pmatrix} I & 0 \\ P & I \end{pmatrix}$
σ_i	R_1	R_2	R_3
$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$	$R_4 = U\alpha U^{-1}$	$R_5 = I$	$R_6 = I$
$\begin{pmatrix} I & \varepsilon_{ij}+\varepsilon_{ji} \\ 0 & I \end{pmatrix}$	$R_7 = \alpha$	$R_8 = I$	R_9
$\begin{pmatrix} I & 0 \\ \varepsilon_{ij}+\varepsilon_{ji} & I \end{pmatrix}$	$R_{10} = \alpha$	R_{11}	$R_{12} = I$

The only non-trivial cases are

a) R_1 is obtained from α by replacing α_{ii} by $\pm\,\delta_{ii}$ and all the other elements in the i-th row and column by 0. Then $|\,R_1\,| = \pm\,\delta_{ii}A_{ii}$, where $A_{ii} = \delta_{ii}\,|\,\alpha\,|$ (since $\delta^T = \alpha^{-1}$) which means that $|\,R_1\,| = \pm\,\delta_{ii}^2\,|\,\alpha\,| = \pm\,|\,\alpha\,|$ (we are computing in Z_8'.)

b) $R_2\,(R_3)$ has non-zero non-diagonal elements only in the i-th row (column) and all the diagonal elements are $\pm$ 1, therefore $|\,R_2\,| = |\,R_3\,| = \pm\,1$.

c) To evaluate R_9 one uses Lemma 4.2. We have $R_9 = I + 2B$, where $B = (\varepsilon_{ij}+\varepsilon_{ji})\,Q$. For $i \neq j$, B is a matrix with only two non-zero rows and $Q = \frac{1}{2}P = \varphi + \varphi^T$. If $Q = (q_{st})$, we verify that for $i \neq j$

$$SpB = 2q_{ij} \tag{4.2}$$

and, with the notation of Lemma 4.2,

$$B_{kl} = 0 \qquad (k, l) \neq (i,j), \quad B_{ij} = q_{ij}^2 - q_{ii}q_{jj} \equiv q_{ij}^2 \mod 2 \tag{4.3}$$

(we are using the fact that Q is symmetric and has even diagonal elements).

By (4.2) and (4.3),

$$|\,R_9\,| = I + 4q_{ij} + 4q_{ij}^2 = 1$$

since $q_{ij} \equiv q_{ij}^2 \mod 2$.

If $i = j$, $B = 2\,\varepsilon_{ii}\,Q$ and $Sp\,B = 2q_{ii} \equiv 0 \bmod 4$ while B_{kl} are zero for all $k < l$, so that $|\,R_9\,| = 1 + 2\,Sp\,B = 1$.

d) The case of R_{11} is similar to that of R_9. Again, $|\,R_{11}\,| = 1$.

PROPOSITION 4.4. *c is a homomorphism and vanishes on $RU(\Gamma)$.*

Proof. Since reduction mod 8 maps $SU_r(Z)$ and $SU_r(Z, 2Z)$ into $SU_r(Z_8)$ and $SU_r(Z_8, Z_4)$, Lemma 4.4 implies that

(4.4) $\qquad \kappa(ASA^{-1}) = \kappa(S)$ for $A \in SU_r(Z)$ and $S \in SU_r(Z, 2Z)$.

Let $A = (A_1, A_2)$, $B = (B_1, B_2) \in SU(\Gamma)$; by (4.4) and Lemma 4.3

$$c(AB) = \pm\,\kappa(A_1B_1B_2^{-1}A_2^{-1}) = \pm\,\kappa(A_1^{-1}(A_1B_1B_2^{-1}A_2^{-1})\,A_1) =$$
$$= \pm\,\kappa(B_1B_2^{-1})\,\kappa(A_2^{-1}A_1) = c(B)\,\kappa(A_1A_2^{-1}A_1A_1^{-1}) = c(B)\,c(A)\,.$$

We have, of course, used the fact that $A_i \in SU_r(Z)$ and that $A_1B_1B_2^{-1}A_1^{-1}$ etc. belong to $SU_r(Z, 2Z)$.

Since c is now a homomorphism, the second assertion follows by checking it on the generators of $RU_r(\Gamma)$, which is entirely trivial.

THEOREM 4.5. *For k odd c defines an isomorphism of $L_{2k+1}^{+}(Z_2) = SU(\Gamma)/RU(\Gamma)$ onto Z_2.*

Proof. In view of Propositions 3.3 and 4.4 it is enough to exhibit an element D in $SU(\Gamma)$ for which c is non-trivial. Such an element is indeed $D = (D_1, I)$, where D_1 is described by (3.2), since $(D_1) = -3$ which is not congruent to ± 1 mod 8.

COROLLARY 4.6. *Let k be odd and let $L_{2k+1}^{+}(Z)$ be the orientable Wall group, defined similarly to $L_{2k+1}^{+}(Z_2)$ by considering the ring $Z(Z) = Z[x, x^{-1}]$ with the "trivial" involution. Then $L_{2k+1}^{+}(Z)$ is generated by the class of the matrix*

$$S = \begin{pmatrix} 1+x+x^{-1} & x+x^{-1} \\ -x-x^{-1} & 1-x-x^{-1} \end{pmatrix}$$

and the map $r\colon L_{2k+1}^{+}(Z) \to L_{2k+1}^{+}(Z_2)$ induced by $Z \to Z_2$ is an isomorphism.

Proof. The " trivial " involution $Z(Z) \to Z(Z)$ which corresponds to the orientable case $w = 1$ is of course not really the identity; it maps x into x^{-1}. It is easy to check that S belongs to $SU(Z(Z))$ with respect to this involution. $r(S) \in SU_1(Z(Z_2))$ corresponds under the identification of $Z(Z_2)$ with Γ to the pair (A_1, A_2), where

$$A_1 = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}$$

and the first element of $A_1 A_2^{-1}$ is 5 whence $c(r(S)) = c(A_1, A_2)$ is non-trivial. The result now follows from the fact [2], [6])) that $L_{2k+1}^{+}(Z) = Z_2$ and from Theorem 4.5.

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