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HILBERT MODULAR SURFACES [1]

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§ 0. INTRODUCTION AND PREPARATORY MATERIAL

0.1. In my Tokyo IMU-lectures I began with a survey of the Hilbert modular group G of a totally real field of degree n over the rationals, or more generally of discontinuous groups Γ operating on $\mathfrak{H}^n$ where $\mathfrak{H}$ is the upper half plane. Then I concentrated on the case $n = 2$ and studied the non-singular algebraic surfaces (Hilbert modular surfaces) which arise by passing from $\mathfrak{H}^2/G$ to the compactification $\overline{\mathfrak{H}^2/G}$ and by resolving all singular points of the normal complex space $\overline{\mathfrak{H}^2/G}$. I gave the proof for the resolution of the cusp singularities, a result announced in my Bourbaki lecture [39]. Then I talked about the calculation of numerical invariants (arithmetic genus, signature) of the Hilbert modular surfaces and on the problem of deciding which of these surfaces are rational. This problem is studied in the present paper with much more detail than in the lectures. We construct certain curves on the Hilbert modular surfaces (arising from imbeddings of $\mathfrak{H}$ in $\mathfrak{H}^2$). Properties of the configuration of such curves

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[1]) International Mathematical Union lectures, Tokyo, February-March 1972.

together with the curves coming from the resolution of the cusp singularities imply in some cases that the surfaces are rational. In particular we take the field $K = \mathbf{Q}(\sqrt{p})$, where p is a prime $\equiv 1 \bmod 4$ and investigate the corresponding compact non-singular Hilbert modular surface $Y(p)$ and the surface obtained by dividing $Y(p)$ by the involution T coming from the permutation of the factors of $\mathfrak{H}^2$. The surface $Y(p)/T$ is rational for exactly 24 primes, a result which was not yet known completely when I lectured in Tokyo.

Up to now rationality of the Hilbert modular surface or of its quotient by the involution T was known only for the fields $\mathbf{Q}(\sqrt{2})$, $\mathbf{Q}(\sqrt{3})$, $\mathbf{Q}(\sqrt{5})$, (H. Cohn, E. Freitag ([14], part II), Gundlach [22], Hammond [25], [26]).

In the following section I shall say a few words about further classification results which were mostly proved only after the time of the Tokyo lectures.

0.2. I have learnt a lot from van de Ven concerning the classification of algebraic surfaces; in fact, the rationality for many of the 24 primes was proved jointly using somewhat different methods. The surfaces $Y(p)$ and $Y(p)/T$ are regular, i.e. their first Betti number vanishes. Van de Ven and I (see [41]) used the above mentioned curves to decide how the surfaces $Y(p)$ ($p \equiv 1 \bmod 4$, p prime) fit into the rough classification of algebraic surfaces (see Kodaira [46], part IV). The result is as follows: *the surface $Y(p)$ is rational for $p = 5, 13, 17$, a blown-up $K3$-surface for $p = 29, 37, 41$, a blown-up elliptic surface* (not rational, not $K3$) *for $p = 53, 61, 73$, and of general type for $p > 73$.*

Also the surfaces $Y(p)/T$ ($p \equiv 1 \bmod 4$, p prime) can be studied by the same methods, but here some refined estimates about certain numerical invariants are necessary.

A joint paper with D. Zagier [42] will show that the surfaces are blown-up $K3$-surfaces for the seven primes $p = 193, 233, 257, 277, 349, 389, 397$ and blown-up elliptic surfaces (not rational, not $K3$) for $p = 241, 281$. We do not know what happens for the eleven primes $p = 353, 373, 421, 461, 509, 557, 653, 677, 701, 773, 797$. As indicated before, there are 24 primes for which the surface is rational. Except for these 44 primes (eleven of them undecided) the surface $Y(p)/T$ is of general type.

Unfortunately a report on these classification problems could not be included in this paper. It is already too long. We must refer to [41], [42].

0.3. Our standard reference for the study of discontinuous groups operating on $\mathfrak{H}^n$ is Shimizu's paper [71] where other references are given.

For the general theory of compactification we refer to the paper of Baily and Borel [4] and the literature listed there. They mention in particular the earlier work on special cases by Baily, Pyatetskii-Shapiro [63], Satake and the Cartan seminar [67]. Compare also Christian [11], Gundlach [20]. Borel and Baily refer to similar general theorems found independently by Pyatetskii-Shapiro.

We cite from the introduction of the paper by Baily and Borel:

"This paper is chiefly concerned with a bounded symmetric domain X and an arithmetically defined discontinuous group Γ of automorphisms of X. Its main goals are to construct a compactification V^* of the quotient space $V = X/\Gamma$, in which V is open and everywhere dense, to show that V^* may be endowed with a structure of normal analytic space which extends the natural one on V, and to establish, using automorphic forms, an isomorphism of V^* onto a normally projective variety, which maps V onto a Zariski-open subset of the latter."

Of course, it suffices if X is equivalent to a bounded symmetric domain. We are concerned in this paper with the case $X = \mathfrak{H}^n$. We do not require that Γ be arithmetically defined, but assume that it satisfies Shimizu's condition (F), see 1.5 in the present paper. Also under this assumption the compactification of $\mathfrak{H}^n/\Gamma$ (which we call $\overline{\mathfrak{H}^n/\Gamma}$) is well-defined and is a normally projective variety. The projective imbedding is given again by automorphic forms in the usual manner. (Compare also Gundlach [20] and H. Cartan ([9], [66] Exp. XV). For $n = 2$ we are able to resolve the singularities and obtain from $\overline{\mathfrak{H}^2/\Gamma}$ a non-singular (projective) algebraic surface.

0.4. As far as I know, the resolution (which exists according to Hironaka [34]) of the singularities of V^* (see the above quotation from the paper of Baily and Borel) has been explicitly constructed only in a very few cases: by Hemperly [33], if $X = \{ z \in \mathbf{C}^2 : |z_1|^2 + |z_2|^2 < 1 \}$, in the present paper if $X = \mathfrak{H}^2$ (thus settling the only cases where the complex dimension of V^* equals 2) and by Igusa [43] for some groups Γ acting on the Siegel upper half plane of degree $g \leqq 3$. Two days before writing this introduction (Jan. 27, 1973) I heard that Mumford is working on the general case (Lecture at the Tata Institute, January 1973).

0.5. It is assumed that the reader is familiar with some basic concepts and results of algebraic number theory ([6], [30], [52]), the theory of differentiable manifolds and characteristic classes [36], the theory of algebraic surfaces ([45], [46], [64]) and the resolution of singularities in the 2-dimensional case ([35], [49]). The definitions and theorems needed can be found, for example, in the literature as indicated.

0.6. The "adjunction formula" ([45], Part I) will be used very often. We therefore state it here.

Let X be a (non-singular) complex surface, not necessarily compact. By $e \cdot f$ we denote the intersection number of the integral 2-dimensional homology classes e, f (one of them may have non-compact support). For two divisors E, F (at least one of them compact), $E \cdot F$ denotes the intersection number of the homology classes represented by E and F. Let $c_1 \in H^2(X, \mathbf{Z})$ be the first Chern class of X. The value of c_1 on every 2-dimensional integral homology class of X (with compact support) is well-defined, and for a compact curve D on X we let $c_1[D]$ be the value of c_1 on the homology class represented by D. By $\tilde{D}$ we denote the non-singular model of D and by $e(\tilde{D})$ its Euler number.

Adjunction formula.

Let D be a compact curve (not necessarily irreducible) on the complex surface X. Then

$$e(\tilde{D}) = c_1[D] - D \cdot D + \sum_{\mathfrak{p}} c_{\mathfrak{p}} \tag{1}$$

The sum extends over the singular points of D, and the summand $c_{\mathfrak{p}}$ is a positive even integer for every singular point $\mathfrak{p}$, depending only on the germ of D in $\mathfrak{p}$.

If K is a canonical divisor on X, then its cohomology class equals $-c_1$. We have

$$c_1[D] = -K \cdot D. \tag{2}$$

0.7. We shall use some basic facts on group actions [47].

Definition. A group G acts properly discontinuously on the locally compact Hausdorff space X if and only if for any $x, y \in X$ there exist neighborhoods U of x and V of y such that the set of all $g \in G$ with $gU \cap V \neq \emptyset$ is finite. An *equivalent condition is that, for any compact subsets K_1, K_2 of X, the set of all $g \in G$ with $g(K_1) \cap K_2 \neq \emptyset$ is finite.*

For a properly discontinuous action, the orbit space X/G is a Hausdorff space. For any $x \in X$, there exists a neighborhood U of x such that the (finite) set of all $g \in G$ with $gU \cap U \neq \emptyset$ equals the isotropy group $G_x = \{g \mid g \in G, g(x) = x\}$. If X is a normal complex space and G acts properly discontinuously by biholomorphic maps, then X/G is a normal complex space.

THEOREM. (H. Cartan [8], and [66] Exp. I). *If X is a bounded domain in $\mathbf{C}^n$, then the group $\mathfrak{A}$ of all biholomorphic maps $X \to X$ with the topology of compact convergence is a Lie group. For compact subsets K_1, K_2 of X, the set of all $g \in \mathfrak{A}$ such that $gK_1 \cap K_2 \neq \emptyset$ is a compact subset of $\mathfrak{A}$. A subgroup of $\mathfrak{A}$ is discrete if and only if it acts properly discontinuously.*

If X is a bounded symmetric domain, then a discrete subgroup Γ of $\mathfrak{A}$ operates freely if and only if it has no elements of finite order.

0.8. I wish to express my gratitude to M. Kreck and T. Yamazaki. Their notes of my lectures in Bonn (Summer 1971) and Tokyo (February-March 1972) were very useful when writing this paper. I should like to thank D. Zagier for mathematical and computational help. Conversations and correspondence with H. Cohn, E. Freitag, K.-B. Gundlach, W. F. Hammond, G. Harder, H. Helling, C. Meyer, W. Meyer, J.-P. Serre, A. V. Sokolovski, A. J. H. M. van de Ven (see 0.2) and A. Vinogradov were also of great help.

Last but not least, I have to thank Y. Kawada and K. Kodaira for inviting me to Japan. I am grateful to them and all the other Japanese colleagues for making my stay most enjoyable, mathematically stimulating, and profitable by many conversations and discussions.

§ 1. THE HILBERT MODULAR GROUP AND THE EULER NUMBER OF ITS ORBIT SPACE

1.1. Let $\mathfrak{H}$ be the upper half plane of all complex numbers with positive imaginary part. $\mathfrak{H}$ is embedded in the complex projective line $\mathbf{P}_1\mathbf{C}$. A complex matrix $\binom{a\ b}{c\ d}$ with $ad - bc \neq 0$ operates on $\mathbf{P}_1\mathbf{C}$ by