Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 19 (1973)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HILBERT MODULAR SURFACES
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Kapitel: §3. NUMERICAL INVARIANTS OF SINGULARITIES AND OF HILBERT MODULAR SURFACES
DOI: https://doi.org/10.5169/seals-46292

Example. Let d be a square-free number > 1 and suppose $d \equiv 2 \bmod 4$ or $d \equiv 3 \bmod 4$. The $(\sqrt{d}, 1)$ is an admissible $\mathbf{Z}$-base of the ideal (1) in $\mathfrak{o}_K$ for $K = \mathbf{Q}(\sqrt{d})$. The quadratic form is given by

$$-u^2 d + v^2$$

and has discriminant $4d$. The first root equals $\frac{-\sqrt{d}}{d} = -\frac{1}{\sqrt{d}}$ which is equivalent to $\sqrt{d}$. (Take always the positive square root). The admissible cycle of natural numbers is obtained by developing $\sqrt{d}$ in a continued fraction.

§ 3. Numerical invariants of singularities and of Hilbert modular surfaces

3.1. Let X be a compact oriented manifold of dimension $4k$ with or without boundary. Then $H^{2k}(X, \partial X; \mathbf{R})$ is a finite dimensional real vector space over which we have a bilinear symmetric form B with

$$B(x, y) = (x \cup y)[X, \partial X], \quad \text{for} \quad x, y \in H^{2k}(X, \partial X; \mathbf{R}),$$

where $[X, \partial X]$ denotes the generator of $H_{4k}(X, \partial X; \mathbf{Z})$ defined by the orientation. The signature of B, i.e., the number of positive entries minus the number of negative entries in a diagonalized version, is called sign (X). If X has no boundary and is differentiable, then according to the signature theorem ([36], p. 86)

$$\text{sign}(X) = L_k(p_1, \ldots, p_k)[X], \tag{1}$$

where L_k is a certain polynomial of weight k in the Pontrjagin classes of X with rational coefficients $(p_j \in H^{4j}(X, \mathbf{Z}))$.

Let N be a compact oriented differentiable manifold without boundary of dimension $4k - 1$ together with a given trivialization α of its stable tangent bundle. (Such a trivialization need not exist). We shall associate to the pair (N, α) a rational number $\delta(N, \alpha)$. Since N has a trivial stable tangent bundle, all its Pontrjagin and Stiefel-Whitney numbers vanish. Therefore N bounds a $4k$-dimensional compact oriented differentiable manifold X. By the parallelization α we get from the stable tangent bundle of X an $\mathbf{SO}$-bundle over X/N. We denote its Pontrjagin classes by

$\tilde{p}_j \in H^{4j}(X/N, \mathbf{Z})$. Then the element $L_k(\tilde{p}_1, \ldots, \tilde{p}_k) \in H^{4k}(X/N, \mathbf{Z}) = H^{4k}(X, \partial X; \mathbf{Z})$ is well-defined.

The number $\delta(N, \alpha)$ is defined by the following formula

$$\delta(N, \alpha) = L_k(\tilde{p}_1, ..., \tilde{p}_k)[X, \partial X] - \operatorname{sign}(X) \tag{2}$$

Thus $\delta(N, \alpha)$ is the deviation from the validity of the signature theorem. It follows from the Novikov additivity of the signature ([3], p. 588) that $\delta(N, \alpha)$ does not depend on the choice of X. If N is of dimension $2n - 1$ (n odd), then we put $\delta(N, \alpha) = 0$.

Remark. The invariant $\delta(N, \alpha)$ and similar invariants were studied also by other authors (Atiyah [1], Kreck [48], W. Meyer [57], S. Morita [59]). In [48] the invariant $\delta(N, \alpha)$ was calculated in several cases.

3.2. We now go back to 2.1. For a cusp of type (M, V) with isotropy group $\mathfrak{G}$ (see 2.1. (1)) we have a $(2n-1)$-dimensional manifold N which is a T^n-bundle over T^{n-1} (see 1.5). We can write (for a fixed positive d)

$$N = \partial X, \quad \text{where} \quad X = W(d) / \mathfrak{G}, \qquad \text{and}$$

$$W(d) = \{ z \mid z \in \mathfrak{H}^n, \prod_{j=1}^{n} \operatorname{Im}(z_j) \geqq d \}.$$

Here X is a (non-compact) complex manifold and is canonically parallelized. Namely, it inherits the standard parallelization of $\mathfrak{H}^n$ given by the coordinates $x_1, y_1, ..., x_n, y_n$ (with $z_k = x_k + iy_k$). This parallelization is respected by $\mathfrak{G}$ if we use unit vectors with respect to the invariant metric of $\mathfrak{H}^n$. Thus the stable tangent bundle of N has a canonical parallelization α. We orient N by the orientation induced by the orientation of X. The rational number $\delta(N, \alpha)$ is now defined. We associate it to the cusp and call it $\delta(\mathfrak{G})$ or $\delta(M, V)$ if $\mathfrak{G} = G(M, V)$. Observe that X cannot be used for the calculation of δ according to (2) because it is not compact. If one compactifies X by adding the point ∞, then one would get a compact manifold $\tilde{X}$ with $\partial \tilde{X} = N$ after resolving the singularity at ∞. This manifold $\tilde{X}$ could be used to calculate δ.

We have associated a rational number $\delta(\mathfrak{G})$ to any "cusp" of type (M, V) with isotropy group $\mathfrak{G}$ where M is a complete $\mathbf{Z}$-module of a totally real field K of degree n over $\mathbf{Q}$ and V a subgroup of finite index of U_M^+. If $V = U_M^+$, we write $\delta(M)$ instead of $\delta(M, U_M^+) = \delta(G(M, U_M^+))$.

By definition, $\delta(\mathfrak{G}) = 0$ if n is odd

If we multiply M by $\gamma \in K$, then

$$\delta(\gamma M, V) = \operatorname{sign} N(\gamma) \cdot \delta(M, V)$$

where $N(\gamma) = \gamma^{(1)} \cdot \gamma^{(2)} \cdot \ldots \cdot \gamma^{(n)}$. Namely, the map

$$z_j \mapsto \gamma^{(j)} z_j^{(\gamma)}$$

with $z_j^{(\gamma)} = z_j$ if $\gamma^{(j)} > 0$ and $z_j^{(\gamma)} = \overline{z_j}$ if $\gamma^{(j)} < 0$ induces a diffeomorphism of $W(d) / G(M, V)$ onto $W(|N(\gamma)| \cdot d) / G(\gamma M, V)$ of degree sign $N(\gamma)$ which is compatible with the parallelizations, and it follows from (2) that the invariant changes sign under orientation reversal.

In particular, $\delta(M, V) = 0$ *if there exist a unit* ε *of* K *with* $\varepsilon M = M$ *and* $N(\varepsilon) = -1$.

Problem. Give a number-theoretical formula for $\delta(M, V)$. This problem can be solved for $n = 2$:

THEOREM. *Let* M *be a complete* **Z**-*module of a real quadratic field and* $[U_M^+ : V] = a$, *then*

$$\delta(M, V) = \frac{a}{3}\left[-(b_0 + b_1 + \ldots + b_{r-1}) + 3r\right] \tag{3}$$

where $((b_0, \ldots, b_{r-1}))$ *is the primitive cycle associated to* M, (see 2.5).

Proof. The torus bundle N bounds $\tilde{X}$ which is obtained by resolving the singularity ∞ of $X \cup \infty$ where $X = W(d)/G(M, V)$. The boundary of $W(d)$ is a principal homogeneous space (1.5). Therefore the normal unit vector field of the boundary (defined using the orthogonal structure of the tangent bundle of $\mathfrak{H}^2$ given by the invariant metric of $\mathfrak{H}^2$) has constant coefficients with respect to the parallelization of $\mathfrak{H}^2$. The same holds for the normal unit vector field of $N = \partial \tilde{X}$. By a classical result of H. Hopf we can extend the normal field to a section of the tangent bundle of $\tilde{X}$ admitting finitely many singularities whose number counted with the proper multiplicities equals the Euler number $e(\tilde{X})$. Because this section is constant on the boundary with respect to the parallelization, it can be pushed down to a section of the complex vector bundle ξ (fibre $\mathbf{C}^2$) over

$\tilde{X}/N$ induced from the parallelization of the tangent bundle of X. Therefore,

$$(4) \qquad e(\tilde{X}) = c_2(\xi)[\tilde{X}, N]$$

where $c_i(\xi) \in H^{2i}(\tilde{X}/N, \mathbf{Z})$ are the Chern classes. The equation (4) follows from the definition of $c_2(\xi)$ by obstruction theory.

We have ([36], Theorem 4.5.1)

$$p_1(\xi) = c_1(\xi)^2 - 2c_2(\xi)$$

and, since $L_1 = p_1/3$,

$$(5) \qquad \delta(M, V) = \frac{1}{3} p_1(\xi)[\tilde{X}, N] - \operatorname{sign}(\tilde{X})$$

$$= \frac{1}{3}(c_1(\xi)^2 [\tilde{X}, N] - 2e(\tilde{X})) - \operatorname{sign}(\tilde{X})$$

By the theorem at the end of 2.5, the manifold $\tilde{X}$ is obtained from $X \cup \infty$ by blowing up ∞ into a cycle of ar rational curves. $\tilde{X}$ has the union of these curves as deformation retract. Thus

$$(6) \qquad e(\tilde{X}) = b_0(\tilde{X}) - b_1(\tilde{X}) + b_2(\tilde{X})$$

$$= 1 - 1 + ar = ar.$$

The intersection matrix of the curves is negative-definite:

$$(7) \qquad \operatorname{sign}(\tilde{X}) = -ar.$$

The cohomology class $c_1(\xi) \in H^2(\tilde{X}, N; \mathbf{Z})$ corresponds by Poincaré duality to an element $z \in H_2(\tilde{X}, \mathbf{Z})$. Let us denote the rational curves of the cycle by S_j ($j \in \mathbf{Z}/ar\mathbf{Z}$). Then z must be an integral linear combination of the S_j which satisfies

$$(8) \qquad z \cdot S_j - S_j \cdot S_j = 2 \qquad (\text{for } ar \geqq 2)$$

$$(8') \qquad z \cdot S_0 - S_0 \cdot S_0 + 2 = 2 \qquad (ar = 1).$$

This follows from the adjunction formula and the information given in 2.4. Since the intersection matrix of the curves of the resolution has non-vanishing determinant, the equations (8) are satisfied by exactly one element z. We obtain that *the first Chern class $c_1(\xi)$ corresponds by Poincaré duality to*

(9) $$z = \sum_{j=0}^{ar-1} S_j$$

Since $c_1(\xi)^2 [X, N] = z \cdot z = -a \sum_{j=0}^{r-1} b_j + 2ar$, formula (3) follows from (5), (6), (7).

3.3. We shall define an invariant φ for certain isolated normal singularities of a complex space of dimension n. In my Tokyo lectures the invariant φ was introduced for $n = 2$ and then generalized to arbitrary n by Morita [59]. Let us first recall that the signature theorem (3.1 (1)) for a compact complex manifold X can be written in terms of the Chern classes

(10) $$\operatorname{sign}(X) = \bar{L}_n(c_1, \ldots, c_n)[X]$$

where $\bar{L}_n$ is a certain polynomical of weight n with rational coefficients in the Chern classes of X, $(c_i \in H^{2i}(X, \mathbf{Z}))$. It is identically zero if n is odd. Let β_n be the coefficient of c_n in $\bar{L}_n$. If n is even ($n = 2k$), then

(11) $$\beta_{2k} = (-1)^k \frac{2^{2k+1}(2^{2k-1} - 1) B_k}{(2k)!}, \quad k \geqq 1$$

where B_k is the k-th Bernoulli number ([36], 1.3 (7) and 1.5 (11)). For n odd, $\beta_n = 0$.

An isolated normal singularity P of a complex space of complex dimension n is called rationally parallelizable if there exists a compact neighborhood U of P containing no further singularities such that the Chern classes of $U - \{P\}$ are torsion classes, i.e. their images in the rational cohomology groups of $U - \{P\}$ vanish. We may assume that ∂U is a $(2n-1)$-dimensional manifold and U the cone over ∂U with P as center. According to Hironaka [34a] the point P can be "blown-up". We obtain a compact complex manifold $\tilde{U}$ which has a boundary as differentiable manifold, namely $\partial \tilde{U} = \partial U$. The Chern classes c_i of $\tilde{U}$ have vanishing images in the rational cohomology of $\partial \tilde{U}$, thus can be pulled back to classes $\tilde{c}_i \in H^{2i}(\tilde{U}, \partial \tilde{U}; \mathbf{Q})$. The Chern numbers $\tilde{c}_{j_1} \cdot \tilde{c}_{j_2} \ldots \tilde{c}_{j_s} [\tilde{U}, \partial \tilde{U}]$ where $j_1 + \ldots + j_s = n$ and $s \geqq 2$ are rational numbers not depending on the pull-back. Therefore, the rational number $\bar{L}_n(\tilde{c}_1, \ldots, \tilde{c}_n)[\tilde{U}, \partial \tilde{U}]$ is well-defined if we replace in this expression $\tilde{c}_n [\tilde{U}, \partial \tilde{U}]$ by the Euler number of $\tilde{U}$. The invariant φ of the isolated normal singular point P is now defined by

(12) $$\varphi(P) = \bar{L}_n(\tilde{c}_1, \ldots, \tilde{c}_n)[U, \partial \tilde{U}] - \operatorname{sign}(\tilde{U})$$

It can be shown (compare [59]) that $\varphi(P)$ does not depend on the resolution. By definition $\varphi(P) = 0$ for n odd.

For a cusp singularity of type (M, V) *the invariants* δ *and* φ *coincide.* This follows from (4) with 2 replaced by n. The proof of (4) remains unchanged for arbitrary n. Of course, X and $\tilde{X}$ in 3.2 play the role of U and $\tilde{U}$ here.

How to calculate φ for a quotient singularity? Let G be the group of p-th roots of unity where p is a natural number. Let $q_1, \ldots, q_n$ be integers which are all prime to p. Then G operates on $\mathbf{C}^n$ by

(13) $$(z_1, \ldots, z_n) \mapsto (\zeta^{q_1} z_1, \ldots, \zeta^{q_n} z_n), \ \zeta^p = 1,$$

and $\mathbf{C}^n/G$ is a normal complex space with exactly one singular point coming from the origin of $\mathbf{C}^n$.

THEOREM. *Let* P *be the quotient singularity defined by* $(p; q_1, \ldots, q_n)$ *where* $(p, q_j) = 1$ *for all* j, *then*

(14) $$\varphi(P) = \frac{\operatorname{def}(p; q_1, \ldots, q_n)}{p} + \frac{\beta_n}{p}$$

where

(15) $$\operatorname{def}(p; q_1, \ldots, q_n) = i^n \sum_{j=1}^{p-1} \cot \frac{\pi q_1 j}{p} \ldots \cot \frac{\pi q_n j}{p}$$

is the cotangent sum arising from the equivariant signature theorem of Atiyah-Bott-Singer ([2], [3]) *and studied in* [38], [79]. Recall that for n odd the cotangent sum (15), the number β_n and the invariant $\varphi(P)$ all vanish.

The proof of (14) was given by Don Zagier and the author for $n = 2$ using the explicit resolution of the singularity ([35], 3.4). For arbitrary n see Morita [59] whose proof uses the equivariant signature theorem and is similar to a proof in [1] concerning a related invariant. It would be interesting to check (14) also for $n > 2$ by an explicit resolution. But, unfortunately, these are not known.

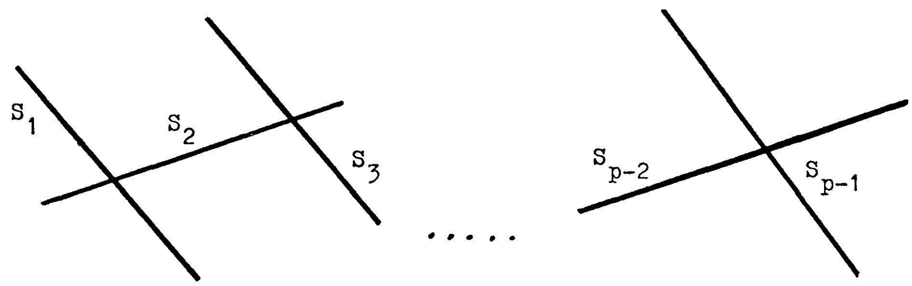
For a quotient singularity P we put

(16) $$\delta(P) = \varphi(P) - \frac{\beta_n}{p} = \frac{\operatorname{def}(p; q_1, \ldots, q_n)}{p}$$

Observe that the δ-invariant in the sense of 3.1 (2) is not defined for a quotient singularity because the boundary N of a neighborhood of such a singularity is a lens space which in general does not admit a parallelization of its stable tangent bundle. However, Atiyah [1] has defined $\delta(N, \alpha)$ by (2) if N is an arbitrary compact oriented differentiable $(4k-1)$-dimensional manifold without boundary and α an integrable connection of the stable tangent bundle of N:

The connection α is extended to a connection α for the stable tangent bundle of X (the extension being taken trivial in a collar of N). Then the Pontrjagin differential forms $\tilde{p}_i$ of $\tilde{\alpha}$ vanish near N and in (2) the value $L_k(\tilde{p}_1, ..., \tilde{p}_k)$ is an integral over a form with compact support in $\mathring{X}$. Again $\delta(N, \alpha)$ does not depend on the choice of X. If one takes in the special case of a quotient singularity for N the lens space and for α the connection inherited from the flat connection on the Euclidean space $\mathbf{R}^{4k} \supset \mathbf{S}^{4k-1}$ $(n = 2k)$ then $\delta(N, \alpha)$ equals the number $\delta(P)$ in (16), (see [1]).

As an example, we calculate $\delta(P)$ if P is the quotient singularity given by $(p; 1, p-1)$. Since $p/(p-1) = [[2, ..., 2]]$ with $p-1$ denominators 2 in the continued fraction, the resolution ([35], 3.4) looks as follows:



where $S_j \cdot S_j = -2$. The adjunction formula implies $\tilde{c}_1 = 0$.

Thus

$$\varphi(P) = \frac{\tilde{c}_1{}^2[\tilde{U}, \partial\tilde{U}] - 2e(\tilde{U})}{3} - \operatorname{sign} U$$

$$= \frac{-2p}{3} + p - 1$$

$$\delta(P) = \varphi(P) + \frac{2/3}{p} = \frac{(p-1)\cdot(p-2)}{3p}$$

Therefore

$$\text{(17)} \qquad \frac{\operatorname{def}(p; 1, p-1)}{p} = \frac{(p-1)\cdot(p-2)}{3p}$$

Let us recall

$$\text{(18)} \qquad \begin{aligned} \operatorname{def}(p; 1, q) &= -\operatorname{def}(p; 1, -q) \\ \operatorname{def}(p; 1, q) &= \operatorname{def}(p; 1, q') \text{ if } qq' \equiv 1 \bmod p \end{aligned}$$

To check the first equation (18) choose the quotient singularity $(p; 1, 1)$. The resolution consists of one curve S_1 with $S_1 \cdot S_1 = -p$. Therefore by the adjunction formula $\tilde{c}_1$ is represented by a homology class $a \cdot S_1$ with

$$a\, S_1 \cdot S_1 - S_1 \cdot S_1 = 2$$

Thus $a = \frac{p-2}{p}$ and $\tilde{c}_1{}^2 [\tilde{U}, \partial \tilde{U}] = -\frac{(p-2)^2}{p}$. We get

$$\begin{aligned} \frac{1}{p} \operatorname{def}(p; 1, 1) &= \frac{1}{3}\left(-\frac{(p-2)^2}{p} - 4 \right) + 1 + \frac{2/3}{p} \\ &= -\frac{1}{3p}(p-1)(p-2) \end{aligned}$$

which checks with (17) and the first equation of (18).

3.4. If Γ is a discrete irreducible subgroup of $(\mathbf{PL}_2^+(\mathbf{R}))^n$ satisfying the condition (F) of the definition in 1.5, then $\mathfrak{H}^n/\Gamma$ has finitely many quotient singularities and no other singularities. It is a rational homology manifold, i.e. every point has a neighborhood which is a cone over a rational homology sphere (in our case a lens space). For $n = 2k$ the signature of $\mathfrak{H}^{2k}/\Gamma$ can be defined using the bilinear symmetric form over $H_{2k}(\mathfrak{H}^{2k}/\Gamma; \mathbf{R})$ given by the intersection number of two elements of this homology group.

In $\mathfrak{H}^{2k}$ we choose around each point z with $|\Gamma_z| > 1$ a closed disk with radius ε measured in the invariant metric and sufficiently small. Then the image of these disks in $\mathfrak{H}^{2k}/\Gamma$ is a finite disjoint union $\bigcup_{\nu=1}^{s} D_{z_\nu}$ where $z_1, \ldots, z_s$ are s points in $\mathfrak{H}^{2k}$ representing the s quotient singularities of $\mathfrak{H}^{2k}/\Gamma$, each D_{z_ν} can be identified with the quotient of the chosen disk around z_ν by the isotropy group Γ_{z_ν}.

Let $x_1, \ldots, x_t$ be a complete set of Γ-inequivalent parabolic points. Choose open sets U_ν as in the definition of 1.5 and denote their images in $\mathfrak{H}^{2k}/\Gamma$ by $D_{x_\nu} = U_\nu/\Gamma_{x_\nu}$. Then

$$\text{(19)} \qquad X = \mathfrak{H}^{2k}/\Gamma - \bigcup_{\nu=1}^{s} \overset{\circ}{D}_{z_\nu} - \bigcup_{\nu=1}^{t} \overset{\circ}{D}_{x_\nu}$$

is a compact manifold with boundary whose signature (as defined in 3.1) equals the signature of $\mathfrak{H}^{2k}/\Gamma$.

THEOREM. *Let Γ be a group of type (F) acting on $\mathfrak{H}^{2k}$. Then*

$$\text{sign}\,(\mathfrak{H}^{2k}/\Gamma) = \sum_{\nu=1}^{s} \delta(z_\nu) + \sum_{\nu=1}^{t} \delta(x_\nu) \tag{20}$$

where $z_1, \ldots, z_s$ are points of $\mathfrak{H}^{2k}$ representing the quotient singularities of $\mathfrak{H}^{2k}/\Gamma$ and $x_1, \ldots, x_t$ is a complete set of Γ-inequivalent parabolic points. For the invariants $\delta(z_\nu)$ see (16). *Recall that the structure of each cusp is determined by a group $\mathfrak{G} \cong \Gamma_{x_\nu}$ (see* 2.1 (1)). *The number $\delta(x_\nu)$ is defined as the number $\delta(\mathfrak{G})$ introduced in* 3.2.

Proof. We first remark that sign $(\mathfrak{H}^{2k}/\Gamma) = 0$ if Γ operates freely and $\mathfrak{H}^{2k}/\Gamma$ is compact. This is a special case of the proportionality of $\mathfrak{H}^{2k}/\Gamma$ and $(\mathbf{P}_1\mathbf{C})^{2k}$, see 1.2, and explains already why (20) does not involve a volume contribution.

Let c_i be the Chern classes of X and $\tilde{c}_i$ pull-backs to the rational cohomology of $X/\partial X$. Then the additivity of the signature and of the Euler number and the validity of the signature theorem for the manifold obtained by resolving all the singularities of the compactification of $\mathfrak{H}^{2k}/\Gamma$ imply

$$\bar{L}_{2k}(\tilde{c}_1, \ldots, \tilde{c}_{2k})[X/\partial X] - \text{sign}\, X + \sum_{\nu=1}^{s} \varphi(z_\nu) + \sum_{\nu=1}^{t} \varphi(x_\nu) = 0 \tag{21}$$

where φ is defined as in 3.3. In $\bar{L}_{2k}(c_1, \ldots, c_{2k})$ we have to interpret $c_{2k}[X/\partial X]$ as Euler number $e(X)$. By § 1 (21)

$$e(X) = \int_{\mathfrak{H}^{2k}/\Gamma} \omega - \sum_{r \geqq 2} a_r(\Gamma)/r$$

The coefficient of c_{2k} in $\bar{L}_{2k}$ equals β_{2k}. Therefore by (21), (16) and because $\varphi(x_\nu) = \delta(x_\nu)$, (see 3.3),

$$\begin{aligned} \text{sign}\, X &= \text{sign}\, \mathfrak{H}^{2k}/\Gamma \\ &= \bar{L}_{2k}(\tilde{c}_1, \ldots, \tilde{c}_{2k-1}, \omega)[X/\partial X] + \sum_{\nu=1}^{s} \delta(z_\nu) + \sum_{\nu=1}^{t} \delta(x_\nu) \end{aligned} \tag{22}$$

where $\omega[X/\partial X]$ has to be interpreted as $\int_{\mathfrak{H}^{2k}/\Gamma} \omega$.

Let d_i be the invariant differential form on $\mathfrak{H}^{2k}$ representing the i-th Chern class in terms of the invariant metric of $\mathfrak{H}^{2k}$. In fact d_i is the

i-th elementary symmetric function of the forms $\omega_j = -\frac{1}{2\pi}\frac{dx_j \wedge dy_j}{y_j^2}$ (see 1.2). The form $\tilde{L}_{2k}(d_1, \ldots, d_{2k})$ is identically 0, because it is a symmetric function in the ω_j^2 which vanish. Recall that $d_{2k} = \omega$. By (22) it remains to show that

$$\tilde{c}_{j_1} \ldots \tilde{c}_{j_s}[X/\partial X] = \int_{\mathfrak{H}^{2k}/\Gamma} d_{j_1} \ldots d_{j_s} \tag{23}$$

for $j_1 + \ldots + j_s = 2k$ and $s \geqq 2$. In the neighborhood of a parabolic point (transformed to ∞) we write

$$\omega_j = d\alpha_j \quad \text{with} \quad \alpha_j = -\frac{1}{2\pi}\frac{dx_j}{y_j}$$

The form α_j is invariant under the isotropy group of the cusp. In the neighborhood of $z_\nu \in \mathfrak{H}^{2k}$ we introduce in each factor of $\mathfrak{H}^{2k}$ geodesic polar coordinates r_j, φ_j with

$$\omega_j = -\frac{1}{2\pi}\sinh(r_j)\, dr_j \wedge d\varphi_j \tag{24}$$

$$\omega_j = d\alpha_j, \quad \text{where} \quad \alpha_j = -\frac{1}{2\pi}(\cosh(r_j) - 1)\, d\varphi_j$$

The form α_j is invariant under the isotropy group Γ_{z_ν}. Take compact manifolds $X''' \subset X'' \subset X' \subset X$ all defined as in (19) and each a compact subset of the interior of the next larger one. We may assume that all the α_j are defined in $\mathfrak{H}^{2k}/\Gamma - X'''$. Choose a C^∞-function ρ which is 0 on X'' and 1 outside X'. Then $\rho\alpha_j$ is a form on $\mathfrak{H}^{2k}/\Gamma$ minus singular points. The form $\omega_j - d(\rho\alpha_j)$ has compact support in X. Thus the elementary symmetric functions in the $\omega_j - d(\rho\alpha_j)$ represent the $\tilde{c}_i$ and the left side of (23) becomes also an integral over $\mathfrak{H}^{2k}/\Gamma$. Recall that the d_i are the elementary symmetric function in the ω_j. By Stokes' theorem the difference of the two sides of (23) is a sum of expressions

$$\lim \int_{\partial D_{x_\nu}} \alpha_j \wedge \omega_1 \wedge \ldots \wedge \hat{\omega}_j \wedge \ldots \wedge \omega_{2k} \tag{25}$$

$$\lim \int_{\partial D_{z_\nu}} \alpha_j \wedge \omega_1 \wedge \ldots \wedge \hat{\omega}_j \wedge \ldots \wedge \omega_{2k} \tag{26}$$

where the limit means that the neighborhoods D_{x_ν} and D_{z_ν} become smaller and smaller, (the number d in 1.5 (16) converges to ∞, the radii of the discs converge to zero). The form in (25) is invariant under the isotropy group of x_ν in the whole group $(\mathbf{PL}_2^+(\mathbf{R}))^{2k}$. Therefore, the integral equals a constant factor times the $(4k-1)$-dimensional volume of ∂D_{x_ν}.

But this volume converges to zero. In (26) for the limit process the integral can be extended over the boundary of a cartesian product of $2k$ discs of radius r divided by Γ_{z_ν}. Let W_r be this cartesian product divided by Γ_{z_ν}. Then

$$|\Gamma_{z_\nu}| \cdot \int_{\partial W_r} \alpha_j \wedge \omega_1 \wedge \dots \wedge \hat{\omega}_j \wedge \dots \wedge \omega_{2k} = (\cosh(r) - 1)^{2k}$$

which converges to zero for $r \to 0$.

3.5. Suppose a cusp is of type (M, V), see 2.1. For $n > 1$ Shimizu ([71], p. 63) associates to the cusp a number $w(M, V)$ which depends only on the strict equivalence class M and the group $V \subset U_M^+$:

Let $(\beta_1, \dots, \beta_n)$ be a base of M. We define

$$d(M) = |\det(\beta_i^{(j)})|.$$

Consider the function

$$L(M, V, s) = \sum_{\mu \in M - \{0\}/V} \frac{\operatorname{sign} N(\mu)}{|N(\mu)|^s} \tag{27}$$

where $N(\mu) = \mu^{(1)} \cdot \mu^{(2)} \cdot \dots \mu^{(n)}$. (The summand in (27) does not change if μ is multiplied with a totally-positive unit. Therefore, it makes sense to sum over the elements of $M - \{0\} / V$.) The function $L(M, V, s)$ can be extended to a holomorphic function in the whole s-plane $\mathbf{C}$. Shimizu defines

$$w(M, V) = \frac{(-1)^{n/2}}{(2\pi)^n} d(M) \cdot L(M, V, 1) \tag{28}$$

We conjecture that also the invariant $\delta(\mathfrak{G})$ (see 3.2) depends only on the pair (M, V). This is clear for $n = 2$. In 3.2 we have defined $\delta(M, V) = \delta(\mathfrak{G})$ if $\mathfrak{G} = G(M, V)$.

The two invariants $\delta(M, V)$ and $w(M, V)$ have similar properties. For example, both vanish if there exists a unit ε of negative norm with $\varepsilon M = M$. Is there a relation between them? A guess would be, I hesitate to say conjecture,

$$2^n w(M, V) = \delta(M, V) \tag{?}$$

This would imply that $w(M, V)$ is always rational. Even this is not known in full generality. However, if M is an ideal in the ring of integers of K, the number $w(M, V)$ is rational. (As Gundlach told me this can be deduced from his paper [24].)

The equation (?) is true for $n = 2$ as we shall see. This was the motivation for Atiyah and Singer to try to relate the invariant δ to L-functions of differential geometry (Lecture of Atiyah at the Arbeitstagung, Bonn 1972). Compare the recent results of Atiyah, Patodi and Singer.

THEOREM. *Let K be a real-quadratic field, M a complete $\mathbf{Z}$-module in K and $V \subset U_M^+$. Then*

$$4\, w(M, V) = \delta(M, V). \tag{29}$$

"*Proof*". Curt Meyer [55] has already studied $w(M, V)$ in 1957. He expressed it in elementary number-theoretical terms using Dedekind sums. It turns out that $\delta(M, V)$ as given in (3) equals Meyer's expression. This will be shown in [42]. Meyer's formula can be found explicitly in [56] (see formulas (6) and (11)) and in Siegel [75] (see formula (120) on p. 183). For more information on the number theory involved we must refer to [42].

3.6. For a non-singular compact connected algebraic surface S the arithmetic genus is defined:

$$\chi(S) = 1 - g_1 + g_2,$$

where g_j is the dimension of the space of holomorphic differential forms of degree j on S. In classical notation $g_1 = q$ and $g_2 = p_g$. The first Betti number of S equals $2g_1$. The numbers g_j are birational invariants. Therefore we can speak of the invariants g_j and of the arithmetic genus of an arbitrary surface possibly with singularities meaning always the corresponding invariant of some non-singular model. We have ([36], 0.1, 0.3)

$$\chi(S) = \frac{1}{12}(c_1^2 + c_2)[S] \tag{30}$$

$$\frac{1}{4}\left(c_2[S] + \frac{1}{3}(c_1^2 - 2c_2)[S]\right),$$

$$\chi(S) = \frac{1}{4}(e(S) + \operatorname{sign}(S)), \tag{31}$$

where $e(S)$ is the Euler number and sign (S) the signature of S. Thus the arithmetic genus is expressed in topological terms, a fact which does not hold in dimensions > 2.

Let Γ be a discrete irreducible group of type (F) acting on $\mathfrak{H}^2$ (see 1.5). The compactification of $\mathfrak{H}^2/\Gamma$ is an algebraic surface. A non-singular model S is obtained by resolving the quotient singularities and the cusp singularities. Then S is a union (glueing along the boundaries) of a manifold X like (19) and of suitable neighborhoods of the configurations of curves into which the singularities were blown up. For every manifold in this union we consider the expression $\frac{1}{4}$ (Euler number + signature). A quotient singularity has a linear resolution ([35], 3.4) and therefore for the neighborhood $\frac{1}{4}(e + \text{sign}) = \frac{1}{4}$, a cusp singularity has a cyclic resolution and therefore $\frac{1}{4}(e + \text{sign}) = 0$ by (6) and (7). The signature and the Euler number behave additively and thus in the notation of (19)

$$\chi(S) = \frac{1}{4}\left(e(X) + \text{sign}(X)\right) + \frac{s}{4}.$$

Since $e(\mathfrak{H}^2/\Gamma) = e(X) + s$, we get

$$\chi(S) = \frac{1}{4}\left(e(\mathfrak{H}^2/\Gamma) + \text{sign}(\mathfrak{H}^2/\Gamma)\right) \tag{32}$$

Using the formulas for $e(\mathfrak{H}^2/\Gamma)$ (see § 1 (21)) and sign (H^2/Γ) (see 20)) we obtain

$$\chi(S) = \frac{1}{4}\int_{\mathfrak{H}^2/\Gamma}\omega + \sum_{\nu=1}^{s}\frac{1}{4}\left(\delta(z_\nu) + (|\Gamma_{z_\nu}| - 1) : |\Gamma_{z_\nu}|\right) + \sum_{\nu=1}^{t}\frac{1}{4}\delta(x_\nu) \tag{33}$$

We have proved the following theorem.

THEOREM. *Let Γ be a discrete irreducible group of type (F) acting on $\mathfrak{H}^2$. Then the arithmetic genus of the compactification $\overline{\mathfrak{H}^2/\Gamma}$ can be expressed by topological invariants of $\mathfrak{H}^2/\Gamma$: Four times the arithmetic genus equals the sum of the Euler number and the signature of $\mathfrak{H}^2/\Gamma$. The arithmetic genus is also given by* (33) *in terms of the Euler volume and contributions coming from the quotient singularities and the cusps.*

Instead of $\chi(S)$ where S is a non-singular model for $\overline{\mathfrak{H}^2/\Gamma}$ we shall write $\chi(\overline{\mathfrak{H}^2/\Gamma})$ or simply $\chi(\Gamma)$. Shimizu ([71], Theorem 11) calculated the dimension of the space $\mathfrak{S}_\Gamma(r)$ of cusp forms of weight r. A cusp form of weight r is defined on $\mathfrak{H}^2$ by a holomorphic form $a(z)(dz_1 \wedge dz_2)^r$ invariant under Γ which vanishes in the cusps. If r is a multiple of all $|\Gamma_{z_\nu}|$, then the Shimizu contributions of the quotient singularities are independent of r and are exactly the contributions which enter in (33). By (29) Shimizu's cusp contributions are exactly the $\dfrac{\delta(x_\nu)}{4}$. Therefore, we can rewrite a special case of Shimizu's result in the following way.

THEOREM. *The assumptions are as in the preceding theorem. Let $r \geqq 2$ be a multiple of all the orders of the isotropy groups of the elliptic fixed points (quotient singularities). Then*

$$\dim \mathfrak{S}_\Gamma(r) = (r^2 - r) \cdot \int_{\mathfrak{H}^2/\Gamma} \omega + \chi(\Gamma) \tag{34}$$

Hence the arithmetic genus of $\overline{\mathfrak{H}^2/\Gamma}$ appears as constant term of the Shimizu polynomial (compare [15], [26]).

Lemma. Let Γ be a discrete irreducible group of type (F) acting on $\mathfrak{H}^2$. The invariant g_1 of the algebraic surface $\overline{\mathfrak{H}^2/\Gamma}$ vanishes. The number $g_2(\overline{\mathfrak{H}^2/\Gamma})$ equals the dimension of the space $\mathfrak{S}_\Gamma(1)$ of cusp forms of weight 1.

"*Proof*". For g_1, see ([14] Teil I, Satz 8) and [26]. For the result on g_2, we have to show that any cusp form of weight 1 can be extended to a holomorphic form θ of degree 2 on the non-singular model obtained by resolving the singularities of $\overline{\mathfrak{H}^2/\Gamma}$. A priori, we have a holomorphic form θ of degree 2 only outside the singularities. It can be extended to the resolution of the quotient singularities ([14], Teil I, Satz 1).

For a cusp singularity the form $\dfrac{du_k \wedge dv_k}{u_k v_k}$ does not depend on the coordinate system (see 2.2 (5)). The form θ is a holomorphic function $f(u_k, v_k)$ multiplied with $\dfrac{du_k \wedge dv_k}{u_k v_k}$. This follows from 2.3 (9) and the remark in 2.5. It is a cusp form if and only if $f(u_k, v_k)$ is divisible by $u_k v_k$. Therefore, θ can be extended.

By the lemma we have

$$(35) \qquad \chi(\Gamma) = 1 + g_2\,\overline{(\mathfrak{H}^2/\Gamma)} = 1 + \dim \mathfrak{S}_\Gamma(1)$$

The group Γ operates also on $\mathfrak{H} \times \mathfrak{H}^-$ where $\mathfrak{H}^-$ is the lower half plane of all complex numbers with negative imaginary part. Since $\mathfrak{H}^2$ and $\mathfrak{H} \times \mathfrak{H}^-$ are equivalent domains, our results are applicable for the action of Γ on $\mathfrak{H} \times \mathfrak{H}^-$. The map $(z_1, z_2) \mapsto (z_1, \overline{z_2})$ induces a homeomorphism

$$(36) \qquad \kappa : \mathfrak{H}^2/\Gamma \to (\mathfrak{H} \times \mathfrak{H}^-)/\Gamma$$

It follows that Γ (as a group acting on $\mathfrak{H} \times \mathfrak{H}^-$) is also of type (F). Because κ is a homeomorphism, the Euler numbers of $(\mathfrak{H} \times \mathfrak{H}^-)/\Gamma$ and $\mathfrak{H}^2/\Gamma$ are equal. Since κ is orientation reversing, we have

$$(37) \qquad \operatorname{sign} (\mathfrak{H} \times \mathfrak{H}^-)/\Gamma = - \operatorname{sign} \mathfrak{H}^2/\Gamma$$

We have denoted the arithmetic genus of $\overline{\mathfrak{H}^2/\Gamma}$ by $\chi(\Gamma)$ and shall put $\chi^-(\Gamma)$ for the arithmetic genus of $\overline{(\mathfrak{H} \times \mathfrak{H}^-)/\Gamma}$. By (32), (35) and (37):

$$(38) \qquad \chi(\Gamma) - \chi^-(\Gamma) = \dim \mathfrak{S}_\Gamma(1) - \dim \mathfrak{S}_\Gamma^-(1) = \tfrac{1}{2} \operatorname{sign} \mathfrak{H}^2/\Gamma,$$

where $\mathfrak{S}_\Gamma^-(1)$ is the space of cusp forms of weight 1 for Γ on $\mathfrak{H} \times \mathfrak{H}^-$.

Remark. The quotient singularities of $\mathfrak{H}^2/\Gamma$ are of the form $(r; 1, q)$. Any such singularity corresponds under κ to a singularity $(r; 1, -q)$. A cusp singularity of type (M, V) goes over into one of type $(\gamma M, V)$ where $N(\gamma) = -1$. Therefore (37) agrees with (20): all contributions coming from the singularities change their sign.

3.7. Let G be the Hilbert modular group for a totally real field K of degree n over $\mathbf{Q}$. The parabolic points are exactly the points of $\mathbf{P}_1 K$ where $\mathbf{P}_1 K$ is regarded as a subset of $(\mathbf{P}_1\mathbf{R})^n$ by the embedding $x \mapsto (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$. The group G acts on $\mathbf{P}_1 K$. The orbits are in one-to-one correspondence with the wide ideal classes of $\mathfrak{o}_K$ (two ideals $\mathfrak{a}, \mathfrak{b}$ are equivalent if there exists an element $\gamma \in K\,(\gamma \neq 0)$ such that $\gamma\mathfrak{a} = \mathfrak{b}$). If $\dfrac{m}{n} \in \mathbf{P}_1 K$ (with $m, n \in \mathfrak{o}_K$) represents an orbit, then $\mathfrak{a} = (m, n)$ represents the corresponding ideal class. *Thus the number of parabolic orbits (cusps) equals the class number h of K.* As in ([75], p. 244) we choose a matrix

$$(39) \qquad A = \begin{pmatrix} m & u \\ n & v \end{pmatrix}, \; mv - nu = 1, u, v \in \mathfrak{a}^{-1}.$$

A simple calculation shows that

(40) $$A^{-1}\,\mathbf{SL}_2(\mathfrak{o}_K)\,A = \mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{a}^2),$$

where, for any ideal $\mathfrak{b} \subset \mathfrak{o}_K$, we set (compare [31])

(41) $$\mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{b}) = \{ \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \mid \alpha\delta - \mathfrak{b}\gamma = 1, \alpha \in \mathfrak{o}_K, \delta \in \mathfrak{o}_K, \beta \in \mathfrak{b}^{-1}, \gamma \in \mathfrak{b} \}$$

Instead of studying the cusp of G at $\frac{m}{n}$, we can consider the cusp of $\mathbf{SL}_2(\mathfrak{o}_K, \mathfrak{a}^2)/\{1, -1\}$ at ∞. Its isotropy group is

$$\{ \left(\begin{smallmatrix}\varepsilon & w\\ 0 & 1/\varepsilon\end{smallmatrix}\right) \mid \varepsilon \in U, w \in \mathfrak{a}^{-2} \} / \{1, -1\} =$$
$$\{ \left(\begin{smallmatrix}\varepsilon^2 & w\\ 0 & 1\end{smallmatrix}\right) \mid \varepsilon \in U, w \in \mathfrak{a}^{-2} \} = G(\mathfrak{a}^{-2}, U^2), \quad \text{see 2.1.}$$

Thus the cusp of G at $\frac{m}{n}$ with $m, n \in \mathfrak{o}_K$ and $(m, n) = \mathfrak{a}$ is given by the pair $(\mathfrak{a}^{-2}, U^2)$.

The extended Hilbert modular group $\hat{G}$ (see 1.7) has the same number of cusps (we have $(\mathbf{P}_1K)/G = (\mathbf{P}_1K)/\hat{G}$). They are given by $(\mathfrak{a}^{-2}, U^+)$.

Let C be the ordinary ideal class group (i.e., the group of wide ideal classes of $\mathfrak{o}_K$) and C^+ the group of narrow ideal classes of $\mathfrak{o}_K$ (with respect to strict equivalence: $\mathfrak{a}, \mathfrak{b}$ are strictly equivalent if there exists a totally positive $\gamma \in K$ with $\gamma\mathfrak{a} = \mathfrak{b}$). Then $\mathfrak{a} \mapsto \mathfrak{a}^{-2}$ induces a homomorphism

(42) $$Sq : C \to C^+.$$

Both G and $\hat{G}$ have h cusps ($h = |C| = h(K)$). The corresponding modules are the squares in C^+, each module occurs k times where k is the order of the kernel of Sq and is a power of 2.

3.8. We consider the Hilbert modular group G and the extended group $\hat{G}$ for $K = \mathbf{Q}(\sqrt{d})$ with d as in 1.4. The cusp singularities of $\overline{\mathfrak{H}^2/G}$ and $\overline{\mathfrak{H}^2/\hat{G}}$ are in one-to-one correspondence with the elements of C. They admit cyclic resolutions. To resolve the cusp belonging to $a \in C$ we take the primitive cycle $((b_0, b_1, \ldots, b_{r-1}))$ associated to $Sq(a) \in C^+$ (see 2.5). This is already the cycle of the resolution if we consider the group $\hat{G}$. For G the cycle of the resolution is $((b_0, b_1, \ldots, b_{r-1}))^c$ where $c = |U^+ : U^2|$.

The cusp at $\infty = \frac{1}{0} \in \mathbf{P}_1K$ has the module $\mathfrak{o}_K$. For $d \equiv 2$ or $3 \bmod 4$ the corresponding primitive cycle is the cycle of the continued fraction for $\sqrt{d}$ (see 2.6). For $d \equiv 1 \bmod 4$ it is the cycle of $\frac{1+\sqrt{d}}{2}$. We list these primitive cycles for those d in the table of 1.7 for which K does not have a unit of negative norm. Also the values of $\delta(\mathfrak{o}_K)$ (see 3.2 (3)) and of the class numbers $h(K)$ are tabulated. If K has a unit of negative norm, then $\delta(\mathfrak{o}_K) = 0$.

d	cycle of $\mathfrak{o}_K$	$\delta(\mathfrak{o}_K)$	$h(K)$
3	((4))	$-\frac{1}{3}$	1
6	((2, 6))	$-\frac{2}{3}$	1
7	((3, 6))	-1	1
11	((2, 2, 8))	-1	1
14	((4, 8))	-2	1
15	((8))	$-\frac{5}{3}$	2
19	((2, 3, 2, 2, 3, 2, 10))	-1	1
21	((5))	$-\frac{2}{3}$	1
22	((4, 2, 2, 2, 4, 10))	-2	1
23	((5, 10))	-3	1
30	((2, 12))	$-\frac{8}{3}$	2
31	((3, 2, 2, 7, 2, 2, 3, 12))	-3	1
33	((2, 3, 2, 7))	$-\frac{2}{3}$	1
34	((6, 12))	-4	2
35	((12))	-3	2
38	((2, 2, 2, 2, 2, 14))	-2	1
39	((2, 2, 2, 14))	$-\frac{8}{3}$	2

3.9. In the next sections we study the signatures of $\mathfrak{H}^2/G$ and $\mathfrak{H}^2/\hat{G}$. Because of (32) this gives also the arithmetic genera $\chi(G)$ and $\chi(\hat{G})$.

THEOREM. *If $K = \mathbf{Q}(\sqrt{d})$ has a unit ε of negative norm, then*

(43) $$\operatorname{sign} \mathfrak{H}^2/G = 0, \quad \chi(G) = \frac{1}{4} e(\mathfrak{H}^2/G).$$

Proof. The actions of G on $\mathfrak{H}^2$ and $\mathfrak{H} \times \mathfrak{H}^-$ are equivalent under $(z_1, z_2) \mapsto (\varepsilon z_1, \varepsilon' z_2)$, (we choose ε positive). The formula (43) follows from (37) and (32).

The following lemma is a corollary of the theorem in 3.4.

Lemma. If K does not have a unit of negative norm, then

(44) $$\operatorname{sign} \mathfrak{H}^2/G = \sum_{\nu=1}^{s} \delta(z_\nu) + 2 \sum_{a \in C} \delta(Sq(a)),$$

(45) $$\operatorname{sign} \mathfrak{H}^2/\hat{G} = \sum_{\nu=1}^{\hat{s}} \delta(\hat{z}_\nu) + \sum_{a \in C} \delta(Sq(a)),$$

Where the points z_ν and $\hat{z}_\nu$ represent the quotient singularities of $\mathfrak{H}^2/G$ and $\mathfrak{H}^2/\hat{G}$ respectively.

The contribution of the quotient singularities in (44) can be calculated using [61], (see 1.7). In [61] not only the orders of the quotient singularities of $\mathfrak{H}^2/G$ are given, but also their types $(r; q_1, q_2)$, see (13). Since $\operatorname{def}(2; 1, 1) = 0$ (see (17)), we only have to consider the quotient singularities of order $r \geqq 3$. For $d \not\equiv 0$ (3) the singularities of order 3 occur in pairs, one of type $(3; 1, 1)$ together with one of type $(3; 1, -1)$. Therefore, their contributions cancel out.

If d is divisible by 3, but $d \neq 3$, we have

(46) $$a_3(G) = 5h\left(\mathbf{Q}(\sqrt{-d/3})\right) \quad \text{for } d \equiv 3 \bmod 9$$
$$a_3(G) = 3h\left(\mathbf{Q}(\sqrt{-d/3})\right) \quad \text{for } d \equiv 6 \bmod 9$$

In the first case $\frac{4}{5}$ of the singularities are of type $(3; 1, 1)$, the others of type $(3; 1, -1)$, in the second case all are of type $(3; 1, 1)$. Therefore, in both cases their contribution in (44) equals (see (17)):

$$3h(\mathbf{Q}(\sqrt{-d/3}))\cdot\frac{1}{3}\operatorname{def}(3;1,1) = -\frac{2}{3}h(\mathbf{Q}(\sqrt{-d/3}))$$

For $d = 3$ there are two singularities of type $(3;1,1)$ and one of type $(6;1,-1)$:

$$d = 3 \Rightarrow \operatorname{sign}\mathfrak{H}^2/G = -2.\frac{2}{9}+\frac{10}{9}-2.\frac{1}{3} = 0$$

We have proved:

THEOREM. *If $K = \mathbf{Q}(\sqrt{d})$ does not have a unit of negative norm, then*

(47) $\operatorname{sign}\mathfrak{H}^2/G = 2\sum_{a\in C}\delta(Sq(a))$ for $d \not\equiv 0 \bmod 3$

$\operatorname{sign}\mathfrak{H}^2/G = 0$ for $d = 3$

$\operatorname{sign}\mathfrak{H}^2/G = -\frac{2}{3}h(\mathbf{Q}(\sqrt{-d/3})) + 2\sum_{a\in C}\delta(Sq(a))$

for $d \equiv 0 \bmod 3$, $d > 3$.

The group C^+ of narrow ideal classes contains the ideal class θ represented by the principal ideals (γ) with $N(\gamma) < 0$. If θ is a square, then

(48) $$2\sum_{a\in C}\delta(Sq(a)) = \sum_{a\in C}\delta(Sq(a)) + \sum_{a\in C}\delta(Sq(a)\,\theta) = 0$$

θ is a square if and only if d is a sum of two squares [25] which happens if and only if d does not contain a prime $\equiv 3 \bmod 4$.

In the contrary case, $\sum_{a\in C}\delta(Sq(a)) < 0$, see [27].

THEOREM. *Let G be the Hilbert modular group for $K = \mathbf{Q}(\sqrt{d})$. Then* $\operatorname{sign}\mathfrak{H}^2/G = 0$ *if and only if $d = 3$ or d does not contain a prime $\equiv 3 \bmod 4$. In all other cases,* $\operatorname{sign}\mathfrak{H}^2/G < 0$.

If the class number of K equals 1, then $\sum_{a\in C}\delta(Sq(a)) = \delta(\mathfrak{o}_K)$. If the class number equals 2 and θ is not a square in C^+, then C^+ is a product of two cyclic groups of order 2 and $\sum_{a\in C}\delta(Sq(a)) = 2\delta(\mathfrak{o}_K)$. Using the tables in 1.7 and 3.8 we have now enough information to calculate the arithmetic genera $\chi(G)$ for $d \leqq 41$. The class numbers $h(\mathbf{Q}(\sqrt{-d/3}))$ which we need for $d = 3, 6, 15, 21, 30, 33, 39$ are 1, 1. 2, 1, 2, 1, 2.

d	$e(\mathfrak{H}^2/G)$	sign $\mathfrak{H}^2/G$	$\chi(G)$	d	$e(\mathfrak{H}^2/G)$	sign $(\mathfrak{H}^2/G)$	$\chi(G)$
2	4	0	1	22	16	−4	3
3	4	0	1	23	18	−6	3
5	4	0	1	26	20	0	5
6	6	−2	1	29	8	0	2
7	6	−2	1	30	24	−12	3
10	8	0	2	31	22	−6	4
11	10	−2	2	33	6	−2	1
13	4	0	1	34	24	0	6
14	12	−4	2	35	28	−12	4
15	12	−8	1	37	8	0	2
17	4	0	1	38	28	−4	6
19	14	—2	3	39	40	−12	7
21	6	−2	1	41	8	0	2

Estimates as in [40] and [42] show that $\chi(G) = 1$ only for finitely many d. Are those in the table the only ones? If d is a prime p, then $\chi(G) = 1$ if and only if $p = 2, 3, 5, 7, 13, 17$ (see 3.12).

The values for sign $\mathfrak{H}^2/G$ are also of interest because (see (38))

$$\dim \mathfrak{S}_G^-(1) - \dim \mathfrak{S}_G(1) = -\frac{1}{2}\operatorname{sign} \mathfrak{H}^2/G \tag{49}$$

Thus $\dim \mathfrak{S}_G^-(1) \geqq \dim \mathfrak{S}_G(1)$, where the inequality is true if and only if d is greater than 3 and divisible by a prime $p \equiv 3 \bmod 4$.

3.10. In view of the preceding theorems it is interesting to calculate $\sum_{a \in C} \delta(Sq(a))$. This was done in [27] for any d using the relation to L-series as explained in 3.5. If d is a prime $\equiv 3 \bmod 4$ the result is especially simple.

THEOREM. *Let p be a prime* $\equiv 3 \bmod 4$ *and* $p > 3$. *Then, for* $K = \mathbf{Q}(\sqrt{p})$, *we have*

$$\sum_{a \in C} \delta(Sq(a)) = -h(-p) \tag{50}$$

Proof. The formulas (27), (28) and (29) imply ([71], p. 69)

$$\sum_{a \in C} \delta(Sq(a)) = \frac{-2}{\pi^2}\sqrt{4p}\, L(1, \chi). \tag{51}$$

Here χ is the unique character with values in $\{1, -1\}$ which is defined for all ideals in $\mathfrak{o}_K$, depends only on the narrow ideal class and satisfies $\chi((\alpha)) = \operatorname{sign} N(\alpha)$ for principal ideals (α).

The function

$$L(s, \chi) = \sum_{\substack{\mathfrak{a} \text{ an ideal} \\ \text{in } \mathfrak{o}_K}} \frac{\chi(\mathfrak{a})}{|N(\mathfrak{a})|^s}$$

can be written as a product

$$L(s, \chi) = L_{-4}(s) L_{-p}(s), \tag{52}$$

where L_{-4} and L_{-p} are the L-functions of $\mathbf{Q}(\sqrt{-4})$ and $\mathbf{Q}(\sqrt{-p})$ over $\mathbf{Q}$. The product decomposition (52) belongs to a decomposition of the discrimant $4p$ of K, namely $4p = (-4)(-p)$, and χ is the genus character corresponding to it ([75], p. 79-80). Evaluating (52) for $s = 1$ implies by a classical formula ([6], V § 4, p. 369)

$$L(1, \chi) = \frac{2\pi}{4} 4^{-1/2} h(-4) \cdot \frac{2\pi}{2} p^{-1/2} h(-p),$$

and this gives (50).

The formula (50) establishes an amusing connection between continued fractions and class numbers. Ordinary continued fractions

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

will be denoted by $[a_0, a_1, a_2, \ldots]$. Let p be a prime $\equiv 3 \bmod 4$. Then ([60], § § 24-26)

$$\sqrt{p} = [a_0, \overline{a_1, a_2, \ldots a_{2s}}], \; a_i \geqq 1, \tag{53}$$

where $a_0 = [\sqrt{p}]$ and $a_{2s} = 2a_0$. The bar over $a_1, a_2, \ldots, a_{2s}$ indicates here the primitive period. The continued fraction development for $\sqrt{p}$ which we needed for the resolution is of the form

$$\sqrt{p} = a_0 + 1 - \cfrac{1}{b_0 - \cfrac{1}{b_1 \cdot_{\cdot_{\cdot}}}} = [[a_0 + 1, \overline{b_0, \ldots, b_{r-1}}]],$$

where the bar indicates again the primitive period. The primitive cycle $((b_0, \ldots, b_{r-1}))$ looks as follows:

$$((\underbrace{2, \ldots, 2}_{a_1 - 1}, a_2 + 2, \underbrace{2, \ldots, 2}_{a_3 - 1}, a_4 + 2, \ldots, \underbrace{2, \ldots, 2}_{a_{2s-1} - 1}, a_{2s} + 2)) \quad .$$

This is shown by an easy calculation (see 2.5 (19)). For $K = \mathbf{Q}(\sqrt{p})$ the signature deviation invariant $\delta(\mathfrak{o}_K)$ is defined (see 3.2 (3)). We have

$$(54) \qquad -3\delta(\mathfrak{o}_K) = \sum_{i=0}^{r-1} (b_i - 3) = \sum_{j=1}^{2s} (-1)^j a_j$$

By (50) and (53) we get:

Proposition. Let p be a prime $\equiv 3 \bmod 4$ *and $p > 3$. Suppose that the class number of $K = \mathbf{Q}(\sqrt{p})$ equals* 1. *Then*

$$(55) \qquad \sum_{j=1}^{2s} (-1)^j a_j = 3h(-p)$$

where $(a_1, a_2, \ldots, a_{2s})$, with $a_{2s} = 2[\sqrt{p}]$, is the primitive period for the ordinary contained fraction development (53) *of $\sqrt{p}$.*

Example. $p = 163$, $h(K) = 1$

$$\sqrt{163} = [12, \overline{1, 3, 3, 2, 1, 1, 7, 1, 11, 1, 7, 1, 1, 2, 3, 3, 1, 24}]$$

$$3h(-163) = 3 \cdot 1 =$$

$$-1+3-3+2-1+1-7+1-11+1-7+1-1+2-3+3-1+24$$

For further information on these and more general number theoretical facts see [42].

3.11. The theorem in 3.10 enables us to give very explicit formulas for the signatures of $\mathfrak{H}^2/G$ and $\mathfrak{H}^2/\hat{G}$ in terms of class numbers of imaginary quadratic fields if $K = \mathbf{Q}(\sqrt{p})$ and p a prime $\equiv 3 \bmod 4$. (For the other primes the signatures vanish).

THEOREM. *Let p be a prime* $\equiv 3 \bmod 4$ *and G the Hilbert modular group* ($\hat{G}$ *the extended one*) *for* $K = \mathbf{Q}(\sqrt{p})$. *Then*

$$\begin{aligned} &\operatorname{sign} \mathfrak{H}^2/G = 0 && \text{for } p = 3 \\ (56) \qquad &\operatorname{sign} \mathfrak{H}^2/G = -2h(-p) && \text{for } p > 3 \\ &\operatorname{sign} \mathfrak{H}^2/\hat{G} = 0 && \text{for } p \equiv 3 \bmod 8 \\ &\operatorname{sign} \mathfrak{H}^2/\hat{G} = -2h(-p) && \text{for } p \equiv 7 \bmod 8 \end{aligned}$$

Proof. The first two equations follow from (47) and (50). For $p > 3$ the quotient singularities of order 3 in $\mathfrak{H}^2/\hat{G}$ occur again in pairs $(3; 1, 1)$, $(3; 1, -1)$ and cancel out in (45). For $p > 3$ and $p \equiv 3 \bmod 8$, there are $h(-p)$ singularities of type $(4; 1, 1)$ and $3h(-p)$ singularities of type $(4; 1, -1)$. For $p \equiv 7 \bmod 8$ there are $2h(-p)$ singularities of type $(4, 1, 1)$, see [61].

The sum of their contributions in (45) equals (see (17))

$$2h(-p)\frac{\operatorname{def}(4; 1, -1)}{4} = h(-p) \qquad \text{for } p \equiv 3 \bmod 8$$

$$2h(-p)\frac{\operatorname{def}(4; 1, 1)}{4} = -h(-p) \qquad \text{for } p \equiv 7 \bmod 8$$

By (45), $\operatorname{sign} \mathfrak{H}^2/\hat{G} = \pm h(-p) - h(-p)$.

It remains to consider the case $p = 3$. We have 3 quotient singularities of order 2, there are 3 others of type $(4; 1, -1)$, $(3; 1, 1)$, $(12; 1, 5)$. By Dedekind-Rademacher reciprocity ([38], (36)) and because $\operatorname{def}(5; 1, 12) = 0$ (see (18))

$$\frac{\operatorname{def}(12; 1, 5)}{12} = 1 - \frac{144 + 1 + 25}{180} = \frac{1}{18}$$

Therefore (see (17) and 3.8):

$$p = 3 \Rightarrow \operatorname{sign} \mathfrak{H}^2/\hat{G} = \frac{1}{2} - \frac{2}{9} + \frac{1}{18} - \frac{1}{3} = 0$$

3.12. For any prime p we know the Euler numbers and the signatures of $\mathfrak{H}^2/G$ and $\mathfrak{H}^2/\hat{G}$. Using 1.6 (21), 3.6 (32) and the theorem of 3.11 we can write down explicit formulas for the arithmetic genera $\chi(G)$ and $\chi(\hat{G})$.

THEOREM. *Let p be a prime $K = \mathbf{Q}(\sqrt{p})$. Let G be the Hilbert modular group for K and $\hat{G}$ the extended one. Then*

$$\chi(G) = 1 \qquad \text{for } p = 2, 3, 5$$
$$\chi(\hat{G}) = 1 \qquad \text{for } p = 3$$

For $p > 5$ we have

$$\chi(G) = \frac{1}{2}\zeta_K(-1) + \frac{h(-4p)}{8} + \frac{1}{6}h(-3p) \qquad \text{for } p \equiv 1 \bmod 4$$

$$\chi(G) = \frac{1}{2}\zeta_K(-1) + \frac{3}{4}h(-p) + \frac{1}{6}h(-12p) \qquad \text{for } p \equiv 3 \bmod 8$$

$$\chi(G) = \frac{1}{2}\zeta_K(-1) + \frac{1}{6}h(-12p) \qquad \text{for } p \equiv 7 \bmod 8$$

$$\chi(\hat{G}) = \frac{1}{4}\zeta_K(-1) + \frac{9}{8}h(-p) + \frac{1}{8}h(-8p) + \frac{1}{12}h(-12p) \quad \text{for } p \equiv 3 \bmod 8$$

$$\chi(\hat{G}) = \frac{1}{4}\zeta_K(-1) + \frac{1}{8}h(-8p) + \frac{1}{12}h(-12p) \qquad \text{for } p \equiv 7 \bmod 8$$

The formulas at the end of 1.3 imply

$$2\zeta_K(-1) = \frac{1}{2}\cdot\pi^{-4} D_K^{3/2}\zeta_K(2) > \frac{1}{2}\pi^{-4} D_K^{3/2}\zeta(4) = \frac{D_K^{3/2}}{180}.$$

It is easy to deduce from this estimate that $\chi(G) = 1$ if and only if $p = 2, 3, 5, 7, 13, 17$ and (for $p \equiv 3 \bmod 4$) $\chi(\hat{G}) = 1$ if and only if $p = 3, 7$. Because of (38) and (56) we also know the arithmetic genera of $\overline{(\mathfrak{H} \times \mathfrak{H}^-)/G}$ and $\overline{(\mathfrak{H} \times \mathfrak{H}^-)/\hat{G}}$ ($p \equiv 3 \bmod 4$). They are equal to 1 if $p = 3$, and both different from 1 if $p > 3$.

§ 4. CURVES ON THE HILBERT MODULAR SURFACES AND PROOFS OF RATIONALITY

We shall construct curves in the Hilbert modular surfaces. They can be used to show that these surfaces are rational in some cases and also for further investigations of the surfaces ([41], [42]). Such curves were studied earlier by Gundlach [23] and Hammond [25]. We need information