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ASYMPTOTIC PROPERTIES OF LINEAR OPERATORS [1]

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1. Positive and regular linear operators.

1.1. One of the most interesting and recent developments in the theory of approximation is a systematic and very successful study of approximation properties of sequences of positive linear operators. One of the best known results in this direction is probably the theorem of Korovkin [25] which states that if a sequence of positive linear operators approximates 1, x and x^2 on $[a, b]$, then it approximates every continuous function on $[a, b]$ (see also [26], pp. 192-196). As we shall see below, this result is typical in a certain sense for positive linear operators. Generally speaking, if positive linear operators have a certain property on a small class of functions, in many cases it can be proved that they have the same property on a larger class of functions. One of the principal aims of this paper is to extend the class of positive linear operators to linear operators which are not necessarily positive but preserve this typical property of positive linear operators.

In classical analysis and especially in the theory of summability there are many examples of positive linear operators, such as the Laplace transform $\mathscr{L}(f, .)$ of an integrable function f on $R^+ = \{x : x \geqq 0\}$, defined by

$$\mathscr{L}(f, x) = \frac{1}{x} \int_0^\infty e^{-(t/x)} f(t)\, dt \qquad (x > 0).$$

One of the basic problems here is to study how well the transform of a function preserves its asymptotic properties. Simplest results of this type for the Laplace transform are:

If f is a bounded, measurable function on R^+, then the Laplace trans-

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form $\mathscr{L}(f, .)$ of f is also a bounded function. If, in addition, $\lim_{x \to \infty} f(x) = \alpha$ then $\lim_{x \to \infty} \mathscr{L}(f, x) = \alpha$.

Results of this type can be easily extended to arbitrary positive linear operators since positive linear operators preserve the inequalities and most of the techniques in asymptotic analysis are based on order properties of R. The results thus obtained are in many respects similar to the theorem of Korovkin. To illustrate this point, let us consider the linear space $\mathscr{F}$ of all real-valued functions on R^+ with the usual order relation $\leqslant$ [1]). Let $\mathscr{F}_0$ be the linear subspace and sublattice of $\mathscr{F}$ consisting of all real-valued functions in $\mathscr{F}$ which are bounded on every finite subinterval of R^+.

We then have the following simple results:

(i) *A positive linear operator* $\Phi: \mathscr{F}_0 \to \mathscr{F}$ *transforms a bounded function into a bounded function if and only if* $\Phi(1, .)$ *is a bounded function on* R^+.

This result follows immediately from the inequality

$$| \Phi(f, x) | \leqq \Phi(|f|, x) \leqq \Phi(1, x) \|f\|$$

where $\|f\| = \sup \{ |f(t)| : t \in R^+ \}$.

(ii) *A positive linear operator* $\Phi: \mathscr{F}_0 \to \mathscr{F}$ *is convergence preserving, i.e.*

$$f \in \mathscr{F}_0 \text{ and } \lim_{t \to \infty} f(t) = c \Rightarrow \lim_{x \to \infty} \Phi(f, x) = c,$$

if $\Phi(1, x) \to 1$ $(x \to \infty)$ *and if there is a positive decreasing function* $g \in \mathscr{F}_0$ *converging to zero as* $x \to \infty$ *such that* $\Phi(g, x) \to 0$ $(x \to \infty)$.

The proof of this result is also very simple. Suppose that $f \in \mathscr{F}_0$ and that $\lim_{t \to \infty} f(t) = c$. Since $\Phi(f, x) - c = \Phi(f - c, x) + c(\Phi(1, x) - 1)$, we have, by positivity of Φ,

$$| \Phi(f, x) - c | \leqq \Phi(|f - c|, x) + |c| \, | \Phi(1, x) - 1 | .$$

In order to estimate the first term on the right-hand side of this inequality, observe that for all $t \in R^+$ we have

$$|f(t) - c| \leqq (\|f\| + |c|) \chi_{[0,\Delta]}(t) + \sup_{t \geqq \Delta} |f(t) - c|$$

$$\leqq \frac{\|f\| + |c|}{g(\Delta)} g(t) + \sup_{t \geqq \Delta} |f(t) - c| .$$

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[1]) $f \leqslant g$ means $f(x) \leqslant g(x)$ for every $x \in R^+$.

Hence

$$\Phi(|f - c|, x) \leqq \frac{\|f\| + |c|}{g(\Delta)} \Phi(g, x) + \sup_{t \geqq \Delta} |f(t) - c| \Phi(1, x),$$

and so

$$|\Phi(f, x) - c| \leqq \frac{\|f\| + |c|}{g(\Delta)} \Phi(g, x) + \sup_{t \geqq \Delta} |f(t) - c| \Phi(1, x)$$
$$+ |c| . |\Phi(1, x) - 1| .$$

Since, by hypotheses, $\Phi(1, x) \to 1$ and $\Phi(g, x) \to 0$ $(x \to \infty)$, it follows that

$$\limsup_{x \to \infty} |\Phi(f, x) - c| \leqq \sup_{t \geqq \Delta} |f(t) - c|,$$

and the result is proved since Δ can be chosen arbitrarily large.

1.2. The most general linear transformations in the theory of summability are not necessarily positive. If $\psi(x, .)$ is a Lebesgue integrable function on R^+ for every fixed $x \in R^+$, and if $\mathscr{S}$ is the family of all measurable functions f on R^+ such that

$$\int_0^\infty |\psi(x, t)| \, |f(t)| \, dt < \infty \text{ for every } x \in R^+,$$

then $\mathscr{S}$ is a linear space and we can define a linear operator G on $\mathscr{S}$ by

$$G(f, x) = \int_0^\infty \psi(x, t) f(t) \, dt . \tag{1.1}$$

We shall consider here, in particular, the subspace $\mathscr{S}_0$ of $\mathscr{S}$, consisting of all functions in $\mathscr{F}$ which are bounded on finite subintervals of R^+.

The classical results of H. Hahn [1] and H. Raff [2], [3] give necessary and sufficient conditions for the operator G to transform every bounded function in $\mathscr{S}$ into an eventually bounded function and a convergent function in $\mathscr{S}_0$ into a convergent function:

A. *In order that, as $x \to \infty$,*

$$f \in \mathscr{S}_0 \text{ and } f(x) = O(1) \quad \Rightarrow \quad G(f, x) = O(1)$$

it is necessary and sufficient that

$$\int_0^\infty |\psi(x, t)| \, dt = O(1) .$$

B. In order that

$$f \in \mathcal{S}_0 \text{ and } \lim_{x\to\infty} f(x) = c \quad \Rightarrow \quad \lim_{x\to\infty} G(f, x) = c$$

it is necessary and sufficient that

(i) $\int_0^\infty \psi(x, t)\, dt \to 1 \quad (x \to \infty)$,

(ii) $\int_0^\infty \psi(x, t)\, \chi_E(t)\, dt \to 0 \quad (x \to \infty)$

for all bounded measurable sets $E \subset R^+$,

(iii) $\int_0^\infty |\psi(x, t)|\, dt = O(1) \quad (x \to \infty)$.

In order to extend the preceding results to more general linear operators, let us observe first that the operator (1.1) can be expressed as a difference of two positive linear operators:

$$G(f, x) = \int_0^\infty \psi^+(x, t) f(t)\, dt - \int_0^\infty \psi^-(x, t) f(t)\, dt, \tag{1.2}$$

where, as usual, $a^+ = \max(a, 0)$, $a^- = -\min(a, 0)$.

It seems therefore that the most natural generalization of the operator (1.1) as well as arbitrary positive linear operators, to operators for which the results of type A and B would be true, should be the class of linear operators which can be expressed as a difference of two positive linear operators. A linear operator which has this property is called a *regular operator*. General theory of regular operators on partially ordered linear spaces can be found in L. V. Kantorovič, B. Z. Vulih, A. G. Pinsker ([5], Ch. VII), B. Z. Vulih ([6], Ch. VIII) and in H. Nakano [7].

For completeness sake, we shall give here an outline of the most important properties of regular operators which will be needed in this paper.

1.3. We shall use here the following definition of regular operators:

Definition: Let $\mathcal{F}_i$, $i = 1, 2$ *be linear subspaces and sublattices of* $\mathcal{F}$. *A linear operator* $\Psi: \mathcal{F}_1 \to \mathcal{F}_2$ *is called a regular operator if there exist positive linear operators* $\Phi_i: \mathcal{F}_1 \to \mathcal{F}$, $i = 1, 2$ *such that* $\Psi(f, x) = \Phi_1(f, x) - \Phi_2(f, x)$ *for every* $f \in \mathcal{F}_1$, *and* $x \in R^+$.

This definition of regular operator does not require the operators Φ_1 and Φ_2 to have values in $\mathscr{F}_2$. In the problems considered here such a requirement is not essential.

We have now the following result.

THEOREM I. *A linear operator $\Psi : \mathscr{F}_1 \to \mathscr{F}_2$ is regular if and only if there exists a positive linear operator $\Phi : \mathscr{F}_1 \to \mathscr{F}$ such that*

$$|\Psi(f, x)| \leqq \Phi(|f|, x) \tag{1.3}$$

for every $f \in \mathscr{F}_1$ and $x \in R^+$.

If Ψ is a regular operator, we have clearly

$$|\Psi(f, x)| \leqq |\Phi_1(f, x)| + |\Phi_2(f, x)|$$

$$\leqq \Phi_1(|f|, x) + \Phi_2(|f|, x) = \Phi(|f|, x)$$

where $\Phi(f, x) = \Phi_1(f, x) + \Phi_2(f, x)$ is a positive linear operator from $\mathscr{F}_1$ into $\mathscr{F}$. Conversely, if (1.3) holds true, let

$$\Phi_1(f, x) = \Phi(f, x),$$

$$\Phi_2(f, x) = \Phi(f, x) - \Psi(f, x).$$

Then $\Psi(f, x) = \Phi_1(f, x) - \Phi_2(f, x)$. Here Φ_1 is obviously a positive linear operator from $\mathscr{F}_1$ into $\mathscr{F}$, and, if $f \geqq 0$ on R^+, we have

$$\Phi_2(f, x) = \Phi(f, x) - \Psi(f, x) \geqq \Phi(f, x) - |\Psi(f, x)| \geqq 0$$

so that Φ_2 is also a positive linear operator from $\mathscr{F}_1$ into $\mathscr{F}$.

In most applications, if a regular operator Ψ is given, it is important to have an intrinsic definition of the operator Φ for which (1.3) is true. This is possible, in view of the following result:

THEOREM II. *A linear operator $\Psi : \mathscr{F}_1 \to \mathscr{F}_2$ is regular if and only if for every $f \in \mathscr{F}_1, f \geqq 0$, and every $x \in R^+$ we have*

$$V_\Psi(f, x) < \infty \tag{1.4}$$

where

$$V_\Psi(f, x) = \sup\{|\Psi(g, x)| : g \in \mathscr{F}_1 \text{ and } |g| \leqq f\}. \tag{1.5}$$

If condition (1.4) is satisfied, we have clearly $V_\Psi(f, .) \in \mathscr{F}$ and the proof that Ψ is regular is carried out in three steps. First, one shows that

V_Ψ is additive on non-negative functions in $\mathscr{F}_1$ and homogeneous with respect to multiplication by positive real numbers:

$$f_1 \geqq 0 \text{ and } f_2 \geqq 0 \Rightarrow V_\Psi(f_1+f_2,x) = V_\Psi(f_1,x) + V_\Psi(f_2,x)\,,$$
$$\alpha > 0 \text{ and } f \geqq 0 \Rightarrow V_\Psi(\alpha f,x) = \alpha V_\Psi(f,x)\,.$$

The operator V_Ψ is then extended to a positive linear operator $\Phi_\Psi\colon \mathscr{F}_1 \to \mathscr{F}$, which coincides with V_Ψ on non-negative functions, in the usual way: if $f = f^+ - f^-$, then $\Phi_\Psi(f,x) = V_\Psi(f^+,x) - V_\Psi(f^-,x)$. Finally, if $f \in \mathscr{F}_1$ we have $|\Psi(f,x)| \leqq V_\Psi(|f|,x) = \Phi_\Psi(|f|,x)$. Hence, by Theorem I, the operator Ψ is regular.

Conversely, if Ψ is regular, by Theorem I, we have a positive linear operator $\Phi\colon \mathscr{F}_1 \to \mathscr{F}$ such that

$$|\Psi(g,x)| \leqq \Phi(|g|,x) \leqq \Phi(f,x)$$

for every $g \in \mathscr{F}_1$, $|g| \leqq f$, and every $x \in R^+$. Hence, the condition (1.4) of Theorem II is satisfied.

If we consider the operator G defined by (1.1), then

$$V_G(f,x) = \int_0^\infty |\psi(x,t)|\, f(t)\, dt\,.$$

From the statement of the theorems of Hahn and Raff mentioned earlier, we can expect that the operator V_Ψ will play an important role in the extension of these results to general regular operators. In fact, as in the theory of positive linear operators, some asymptotic property of the regular operator Ψ will hold for a large class of functions if and only if the operator V_Ψ has certain properties on a much smaller class of functions.

2. Boundedness and convergence preserving regular operators.

In this section and the following one we shall extend to regular operators some of the well-known results about the asymptotic behavior of the special transform G defined by (1.1).

Let us consider the linear space $\mathscr{M}$ of real valued measurable functions on R^+ and let $\mathscr{M}_0$ be the subspace of $\mathscr{M}$ consisting of all measurable functions on R^+ which are bounded on every finite interval of R^+.

The basic result which characterizes regular operators from $\mathscr{M}_0$ into $\mathscr{F}_0$ that preserve boundedness can be stated as follows:

THEOREM 1. *Let* $\Psi : \mathscr{M}_0 \to \mathscr{F}_0$ *be a regular operator. In order that, as* $x \to \infty$,

$$(2.1) \qquad f \in \mathscr{M}_0 \text{ and } f(x) = O(1) \quad \Rightarrow \quad \Psi(f, x) = O(1)$$

it is necessary and sufficient that

$$(2.2) \qquad V_\Psi(1, x) = O(1)$$

where V_Ψ *is defined by* (1.5).

This result is clearly a natural extension of the results for the special operator G mentioned in section *1.2* under A. The corresponding result for regular operators which transform functions in $\mathscr{S}_0$, converging to zero as $x \to \infty$ into bounded functions is given by the following theorem.

THEOREM 2. *Let* $\Psi : \mathscr{M}_0 \to \mathscr{F}_0$ *be a regular operator. In order that, as* $x \to \infty$,

$$(2.3) \qquad f \in \mathscr{M}_0 \text{ and } f(x) \to 0 \quad \Rightarrow \quad \Psi(f, x) = O(1)$$

it is necessary and sufficient that

$$(2.4) \qquad W_\Psi(1, x) = O(1)$$

where W_Ψ *is defined for every* $f \in \mathscr{M}_0, f \geqq 0$, *by*

$$(2.5) \quad W_\Psi(f, x) = \sup \{ | \Psi(g, x) | : g \in \mathscr{M}_0, | g | \leqq f, g = o(f) \} ^{1)} .$$

Condition (2.4) in Theorem 2 is less restrictive than the corresponding condition (2.2) in Theorem 1, since $W_\Psi(f, x) \leqq V_\Psi(f, x)$ for every $f \in \mathscr{M}_0$, $f \geqq 0$ and for every $x \in R^+$.

It is now easy to obtain a generalization of the results for the special operator G mentioned in section *1.2* under *B.*, i.e. to establish necessary and sufficient conditions for a regular operator to be convergence preserving:

THEOREM 3. *Let* $\Psi : \mathscr{M}_0 \to \mathscr{F}_0$ *be a regular operator. In order that, as* $x \to \infty$,

$$(2.6) \qquad f \in \mathscr{M}_0 \text{ and } f(x) \to c \quad \Rightarrow \quad \Psi(f, x) \to c$$

it is necessary and sufficient that

$$(2.7) \qquad \Psi(1, x) \to 1 ,$$

[1] $g = o(f)$ means $g(t) = o(f(t))$ $(t \to \infty)$.

(2.8) $$\Psi(\chi_E, x) \to 0$$

for every bounded measurable subset E of R^+, and

(2.9) $$W_\Psi(1, x) = O(1).$$

3. Transformations of O-regular and slowly varying functions by regular operators.

3.1. The class of positive functions which are eventually bounded away from zero and infinity has been extended to the class of O-regular functions defined as follows:

A positive, measurable function l on R^+ is O-regular if

(3.1) $$\frac{l(\lambda x)}{l(x)} = O(1) \ (x \to \infty)$$

for every $\lambda > 0$.

For example, any function l such that $ax^\alpha \leqq l(x) \leqq Ax^\alpha$, where $\alpha \in R$, clearly satisfies condition (3.1).

The class of O-regular functions and related classes of functions have been studied extensively by V. G. Avakumović [8, 9, 10, 11], J. Karamata [14], N. K. Bari, S. B. Stečkin [15], M. A. Krasnoselskiĭ, T. B. Rutickiĭ [16], W. Matuszewska [17] and others.

The closely related class of slowly varying (SV) functions, introduced by J. Karamata ([12], [13]), generalizes the class of functions converging to a positive limit. A positive, measurable function L defined on R^+ is a slowly varying function if

(3.2) $$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1$$

for every $\lambda > 0$.

Clearly, every measurable function on R^+ which converges to a positive limit as $x \to \infty$ is a SV function. Also, functions like

$$\varphi(x) = \begin{cases} 1, 0 \leqq x < e, \\ \log x, x \geqq e, \end{cases} \quad h(x) = \left(2 + \frac{\sin x}{x}\right) \varphi(x),$$

and their iterations are SV functions. More generally, any measurable function g on R^+ such that $\varphi(x) \leqq g(x) \leqq \varphi(x) + \sqrt{\varphi(x)}$ is a SV function.

The most important properties of O-regular and SV functions can be stated as follows:

Representation Theorems: *If l is an O-regular function, there exist $B > 0$ and bounded measurable functions α and β on $[B, \infty]$ such that*

$$(3.3) \qquad l(x) = \exp\left(\alpha(x) + \int_B^x \frac{\beta(t)}{t}\,dt\right) \text{ for } x \geqq B.$$

If L is a SV function, then for some $B > 0$,

$$(3.4) \qquad L(x) = \exp\left(\eta(x) + \int_B^x \frac{\varepsilon(t)}{t}\,dt\right) \text{ for } x \geqq B,$$

where η and ε are bounded measurable functions on $[B, \infty]$ such that $\eta(x) \to c$ and $\varepsilon(x) \to 0$ $(x \to \infty)$.

A proof of these results for continuous O-regular and *SV* functions can be found in [12], [13], and [14]. These results were subsequently extended to measurable O-regular and *SV* functions by a number of authors (see [18] for details).

One of the typical and simplest results about the asymptotic behavior of special linear transforms of *SV* functions is probably the following result of K. Knopp [19]:

If L is a *SV* function, and if $L \in \mathscr{M}_0$, then

$$\frac{1}{xL(x)} \int_0^\infty e^{-(t/x)} L(t)\,dt \to 1 \quad (x \to \infty).$$

Similar results involving more or less special transformations have been obtained by G. H. Hardy and W. W. Rogosinski [4], S. Aljančić, R. Bojanić, M. Tomić [20], R. Bojanić and J. Karamata [21], and, in slightly different form, by D. Drasin ([22], Th. 6). The most general result of this type, obtained by M. Vuilleumier [23], [24], can be stated as follows:

Let G be defined by (1.1). *In order that*

$$\frac{G(L, x)}{L(x)} \to 1 \quad (x \to \infty)$$

holds for every SV function $L \in \mathscr{M}_0$ it is necessary and sufficient that, as $x \to \infty$,

(i) $\int_0^\infty \Psi(x, t)\,dt \to 1$,

(ii) *there exists* $\eta > 0$ *such that*

$$\int_0^x |\Psi(x,t)|\, t^{-\eta}\, dt = O(x^{-\eta}) \text{ and } \int_x^\infty |\Psi(x,t)|\, t^{\eta} dt = O(x^{\eta}).$$

3.2. Theorem 1 characterizes boundedness preserving operators. A natural extension of that result is the theorem which characterizes regular operators Ψ with the property that $\Psi(l, x) = O(l(x))$ $(x \to \infty)$ holds for every O-regular function $l \in \mathcal{M}_0$. In this direction we have the following result:

THEOREM 4. *Let* $\Psi: \mathcal{M}_0 \to \mathcal{F}_0$ *be a regular operator. In order that*

$$\Psi(l, x) = O(l(x)) \quad (x \to \infty), \tag{3.5}$$

holds for every O-regular function $l \in \mathcal{M}_0$ it is necessary and sufficient that for all $\alpha > 0$, as $x \to \infty$,

$$V_\Psi(t^\alpha, x) = O(x^\alpha) \tag{3.6}$$

and

$$V_\Psi(\chi_{[0,1]}(t) + t^{-\alpha} \chi_{(1,\infty)}(t), x) = O(x^{-\alpha}) \tag{3.7}$$

where V_Ψ is defined by (1.5).

Likewise, as an analog of Theorem 2, the following theorem characterizes regular operators which have the property that

$$\Psi(L, x) = O(L(x)) \quad (x \to \infty)$$

holds for every SV function $L \in \mathcal{M}_0$:

THEOREM 5. *Let* $\Psi: \mathcal{M}_0 \to \mathcal{F}_0$ *be a regular operator. In order that*

$$\Psi(L, x) = O(L(x)) \quad (x \to \infty) \tag{3.8}$$

holds for every SV function $L \in \mathcal{M}_0$ it is necessary and sufficient that there exists $\eta > 0$ such that, as $x \to \infty$,

$$W_\Psi(t^\eta, x) = O(x^\eta) \tag{3.9}$$

and

$$W_\Psi(\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,\infty)}(t), x) = O(x^{-\eta}) \tag{3.10}$$

where W_Ψ is defined by (2.5).

Finally, the analog of Theorem 3 can be stated as follows:

THEOREM 6. *Let* $\Psi : \mathcal{M}_0 \to \mathcal{F}_0$ *be a regular operator. In order that*

(3.11) $$\frac{\Psi(L,x)}{L(x)} \to 1 \quad (x \to \infty)$$

holds for every SV function $L \in \mathcal{M}_0$ *it is necessary and sufficient that*

(3.12) $$\Psi(1,x) \to 1 \quad (x \to \infty),$$

and that the asymptotic relations (3.9) *and* (3.10) *hold for some* $\eta > 0$.

4. PROOFS.

4.1. Proof of Theorem 1. The sufficiency of condition (2.2) follows from the inequality

$$|\Psi(f,x)| \leqq V_{\Psi}(1,x) \; \|f\| .$$

The necessity of (2.2) is proved by way of contradiction. Suppose that (2.2) is not satisfied. Then

(4.1.1) $$\limsup_{x \to \infty} V_{\Psi}(1,x) = \infty .$$

In view of (4.1.1), (2.1) and the properties of Ψ, it is possible to find by induction an increasing sequence (x_k) going to infinity and a sequence (g_k) of functions in $\mathcal{M}_0$ such that, if A_k is defined by $A_k = V_{\Psi}(1, x_k)$, then

(4.1.2) $$A_1 \geqq 16 \text{ and } A_k \geqq 16\, A_{k-1}, \quad k = 2, 3, \ldots,$$

(4.1.3) $$A_k \geqq 16 \,(\sup_{x \varepsilon R^+} | \Psi \,(\sum_{i=1}^{k-1} \frac{g_i}{\sqrt{A_i}}, x) |)^2, \quad k = 2, 3, \ldots,$$

and

(4.1.4) $$|g_k| \leqq 1, \; |\Psi(g_k, x_k)| \geqq \frac{3}{4} A_k, \quad k = 1, 2, \ldots$$

Let

(4.1.5) $$g(x) = \sum_{i=1}^{\infty} \frac{g_i(x)}{\sqrt{A_i}}.$$

By (4.1.2) and (4.1.4), this series is uniformly convergent and consequently g is in $\mathcal{M}$. Also, g is bounded on R^+ since

$$|g(x)| \leqq \sum_{i=1}^{\infty} \frac{|g_i(x)|}{4^i} \leqq \frac{1}{3}.$$

We shall show now that

$$(4.1.6) \qquad |\Psi(g, x)| \to \infty \quad (x \to \infty),$$

which is impossible by (2.1). Hence, (2.2) must be satisfied.

From the definition of g follows that

$$|\Psi(g, x_k)| \geqq \frac{|\Psi(g_k, x_k)|}{\sqrt{A_k}} - \left| \Psi\left(\sum_{i=1}^{k-1} \frac{g_i}{\sqrt{A_i}}, x_k\right) \right| - \left| \Psi\left(\sum_{i=k+1}^{\infty} \frac{g_i}{\sqrt{A_i}}, x_k\right) \right|.$$

By (4.1.3) we have

$$\left| \Psi\left(\sum_{i=1}^{k-1} \frac{g_i}{\sqrt{A_i}}, x_k\right) \right| \leqq \frac{\sqrt{A_k}}{4}.$$

Finally, by (4.1.4) and (4.1.2)

$$\left| \sum_{i=k+1}^{\infty} \frac{g_i(t)}{\sqrt{A_i}} \right| \leqq \sum_{i=k+1}^{\infty} \frac{1}{\sqrt{A_i}} \leqq \frac{1}{\sqrt{A_k}} \sum_{i=k+1}^{\infty} \frac{1}{4^{i-k}} \leqq \frac{1}{3\sqrt{A_k}}.$$

Since Ψ is a regular operator, it follows that

$$\left| \Psi\left(\sum_{i=k+1}^{\infty} \frac{g_i}{\sqrt{A_i}}, x_k\right) \right| \leqq \frac{1}{3\sqrt{A_k}} V_{\Psi}(1, x_k) = \frac{1}{3}\sqrt{A_k}.$$

From these inequalities follows that

$$|\Psi(g, x_k)| \geqq \frac{3}{4}\sqrt{A_k} - \frac{1}{4}\sqrt{A_k} - \frac{1}{3}\sqrt{A_k} = \frac{1}{6}\sqrt{A_k} \geqq \frac{1}{6} 4^k,$$

and (4.1.6) is proved.

The arguments used here are essentially the same as the ones in the proof of Nakano's Theorem [6, Ch. IX] that the limit of a sequence of regular functionals is a regular functional.

4.2. Proof of Theorem 2. The proof of Theorem 2 is quite similar. The sufficiency of condition (2.4) follows from the inequality

$$|\Psi(f,x)| \leqq W_\Psi(1,x)\,\|f\|\,.$$

The necessity of condition (2.4) is proved by way of contradiction. If (2.4) is not satisfied, it is possible to construct by induction an increasing sequence (x_k) going to infinity and a sequence (g_k) of functions in $\mathscr{M}_0$ such that, if A_k is defined by $A_k = W_\Psi(1, x_k)$, the inequalities (4.1.2), (4.1.3) and (4.1.4) are satisfied and moreover

$$|g(x)| < \frac{1}{2^k}, \text{ for all } x \geqq x_k, \quad k = 2, 3, \ldots$$

and

$$g_k(x) \to 0 \quad (x \to \infty)\,.$$

The function g defined by (4.1.5) has then the properties

$$g(x) \to 0 \quad (x \to \infty)$$

and

$$|\Psi(g, x_k)| \to \infty \quad (k \to \infty)\,.$$

This contradicts hypothesis (2.3) and the necessity of condition (2.4) is proved.

4.3. *Proof of Theorem 3.* (Sufficiency). We have

$$|\Psi(f,x) - c| \leqq |\Psi(f-c,x)| + |c|\,.\,|\Psi(1,x) - 1|\,.$$

Given $\varepsilon > 0$, let X_ε be such that $|f(t) - c| \leqq \varepsilon$ for all $t \geqq X_\varepsilon$ and let

$$g_1(t) = (f(t) - c)\,\chi_{[0,X_\varepsilon]}(t)\,,$$
$$g_2(t) = (f(t) - c)\,\chi_{(X_\varepsilon,\infty)}(t)\,.$$

We then have

$$|\Psi(f - c, x)| \leqq |\Psi(g_1, x)| + |\Psi(g_2, x)|\,.$$

Hence,

$$|\Psi(f,x) - c| \leqq |\Psi(g_1,x)| + |\Psi(g_2,x)| + |c|\;|\Psi(1,x) - 1)|\,. \tag{4.3.1}$$

First, we have $|g_2(t)| \leqq \varepsilon$ for every $t \in R^+$ and $g_2 = o(1)$. Hence, by definition of W_Ψ,

$$|\Psi(g_2,x)| \leqq \varepsilon\, W_\Psi(1,x)\,. \tag{4.3.2}$$

Next, we can find a simple function $h = \sum_{i=1}^{N} A_i \chi_{E_i}$, where E_i, $i = 1,\ldots, N$ are measurable subsets of $[0, X_\varepsilon]$, such that

$$|h(t)| \leqq \|g\| \chi_{[0,X_\varepsilon]}(t) \text{ and } \|g - h\| < \varepsilon .$$

Then

$$|\Psi(g_1, x)| \leqq |\Psi(g_1 - h, x)| + |\Psi(h, x)| \tag{4.3.3}$$
$$\leqq \varepsilon W_\Psi(1, x) + \sum_{i=1}^{N} |A_i| \, |\Psi(\chi_{E_i}, x)| .$$

From (4.3.1), (4.3.2), (4.3.3) and the hypotheses (2.7) and (2.8) follows finally that

$$\limsup_{x\to\infty} |\Psi(f, x) - c| \leqq 2\varepsilon \| W_\Psi(1, .)\|$$

and Theorem 1 is proved since ε can be chosen arbitrarily small.

(Necessity). The necessity of condition (2.9) follows from Theorem 2. The necessity of conditions (2.7) and (2.8) is obvious.

4.4. Proof of Theorem 4. (Sufficiency). Let l be any O-regular function in $\mathcal{M}_0$. Define p_α and q_α by

$$p_\alpha(x) = \sup_{0\leqq t\leqq x} l(t)\big(\chi_{[0,1]}(t) + t^\alpha \chi_{(1,\infty)}(t)\big) \tag{4.4.1}$$

and

$$q_\alpha(x) = \sup_{t\geqq x} l(t)\, t^{-\alpha} , \tag{4.4.2}$$

Then it can be shown, using representation (3.3), that there exists $\alpha > 0$ such that

$$p_\alpha(x) = O\big(x^\alpha l(x)\big) \quad (x \to \infty) \tag{4.4.3}$$

and

$$q_\alpha(x) = O\big(x^{-\alpha} l(x)\big) \quad (x \to \infty) . \tag{4.4.4}$$

To show that (3.5) is satisfied, we start with the inequality

$$|\Psi(l, x)| \leqq |\Psi(l\chi_{[0,x]}, x)| + |\Psi(l\chi_{(x,\infty)}, x)| . \tag{4.4.5}$$

First we have by (4.4.1)

$$l(t)\,\chi_{[0,x]}(t)$$
$$= l(t)\big(\chi_{[0,1]}(t) + t^{\alpha}\chi_{(1,\infty)}(t)\big)\,\chi_{[0,x]}(t)\big(\chi_{[0,1]}(t) + t^{-\alpha}\chi_{(1,\infty)}(t)\big)$$
$$\leqq p_{\alpha}(x)\big(\chi_{[0,1]}(t) + t^{-\alpha}\chi_{(1,\infty)}(t)\big)$$

for all $t \geqq 0$. Likewise, by (4.4.2), we have

$$l(t)\,\chi_{(x,\infty)}(t) \leqq q_{\alpha}(x)\,t^{\alpha}$$

for all $t \geqq 0$. By definition of V_{Ψ} and (4.4.5), it follows then that

$$|\,\Psi(l,x)\,| \leqq p_{\alpha}(x)\,V_{\Psi}\big(\chi_{[0,1]}(t) + t^{-\alpha}\,\chi_{(1,\infty)}(t), x\big) + q_{\alpha}(x)\,V_{\Psi}(t^{\alpha}, x)\,.$$

Hence

$$\frac{1}{l(x)}\,|\,\Psi(l,x)\,| \leqq \left(\frac{p_{\alpha}(x)}{x^{\alpha}\,l(x)}\right) x^{\alpha}\,V_{\Psi}\big(\chi_{[0,1]}(t) + t^{-\alpha}\,\chi_{(1,\infty)}(t), x\big)$$
$$+ \left(\frac{q_{\alpha}(x)}{x^{-\alpha}\,l(x)}\right) x^{-\alpha}\,V_{\Psi}(t^{\alpha}, x)\,,$$

and (3.5) follows from (4.4.3), (4.4.4) and hypotheses (3.6) and (3.7).

(Necessity). Let $\alpha > 0$ and let $f \in \mathscr{M}_0$ be a bounded function on R^+. Let

$$g(x) = (2\,\|f\| + f(x))\,x^{\alpha}\,.$$

Then g is an O-regular function, and

$$\frac{1}{x^{\alpha}}\,\Psi\big(f(t)\,t^{\alpha}, x\big) = \frac{1}{x^{\alpha}}\,\Psi(g, x) - \frac{2\,\|f\|}{x^{\alpha}}\,\Psi(t^{\alpha}, x)$$
$$= (2\,\|f\| + f(x))\,\frac{\Psi(g,x)}{g(x)} - \frac{2\,\|f\|}{x^{\alpha}}\,\Psi(t^{\alpha}, x)\,.$$

Hence, by (3.5), we have

$$\Psi\big(f(t)\,t^{\alpha}, x\big) = O(x^{\alpha}) \quad (x \to \infty)\,,$$

for every bounded function f in $\mathscr{M}_0$. Thus, the regular operator Ψ_{α} defined by

$$\Psi_{\alpha}(f, x) = \frac{1}{x^{\alpha}}\,\Psi\big(f(t)\,t^{\alpha}, x\big)$$

transforms every bounded function in $\mathscr{M}_0$ into a bounded function. By Theorem 1, it follows that

$$\text{(4.4.6)} \qquad V_{\Psi_{\alpha}}(1, x) = O(1) \quad (x \to \infty)\,.$$

But given any $g \in \mathscr{M}_0$ such that $|g(t)| \leq t^\alpha$, we have

$$|\Psi(g,x)| = \left|\Psi\left(\frac{g(t)}{t^\alpha} t^\alpha, x\right)\right| = x^\alpha \left|\Psi_\alpha\left(\frac{g(t)}{t^\alpha}, x\right)\right| \leq x^\alpha V_{\Psi\alpha}(1,x).$$

Hence, the supremum of the left hand side over all $g \in \mathscr{M}_0$ such that $|g(t)| \leq t^\alpha$ must satisfy the same inequality:

$$|V_\Psi(t^\alpha, x)| \leq x^\alpha V_{\Psi\alpha}(1,x)$$

and (3.6) follows by (4.4.6).

The proof of (3.7) is similar to that of (3.6) except that the function t^α, $\alpha > 0$, has to be replaced in the argument by the function $\chi_{[0,1]}(t) + t^{-\alpha}\chi_{(1,\infty)}(t)$.

4.5. *Proof of Theorem 5.* (Sufficiency). Given any *SV* function $L \in \mathscr{M}_0$ and any $\eta > 0$, let

$$P_\eta(x) = \sup_{0 \leq t \leq x} t^\eta L(t)$$

and

$$Q_\eta(x) = \sup_{t \geq x} t^{-\eta} L(t).$$

Then

$$\frac{P_\eta(x)}{x^\eta L(x)} \to 1 \quad (x \to \infty) \tag{4.5.1}$$

and

$$\frac{Q_\eta(x)}{x^{-\eta} L(x)} \to 1 \quad (x \to \infty). \tag{4.5.2}$$

The proofs of these relations for continuous *SV* functions can be found in [12] and [13]. For measurable *SV* functions, the proofs follow easily from the representation theorem.

Clearly, if P_η is defined by

$$P_\eta(x) = \sup_{0 \leq t \leq x} \left(\chi_{[0,1]}(t) + t^\eta \chi_{(1,\infty)}(t)\right) L(t), \tag{4.5.3}$$

it will have again the property (4.5.1).

To prove that (3.8) is satisfied, we start with the inequality

$$|\Psi(L,x)| \leq |\Psi(L\chi_{[0,x]}, x)| + |\Psi(L\chi_{(x,\infty)}, x)|. \tag{4.5.4}$$

First we have by (4.5.3),

$$L(t)\,\chi_{[0,x]}(t)$$
$$= (\chi_{[0,1]}(t) + t^{\eta}\chi_{(1,\infty)}(t))\,L(t)\,\chi_{[0,x]}(t)\,(\chi_{[0,1]}(t) + t^{-\eta}\,\chi_{(1,\infty)}(t))$$
$$\leqq P_{\eta}(x)\,(\chi_{[0,1]}(t) + t^{-\alpha}\,\chi_{(1,\infty)}(t))$$

for all $t \geqq 0$. Since

$$L(t)\,\chi_{[0,x]}(t) = o\,(t^{-\eta}) \quad (t \to \infty),$$

it follows, by definition of W_{Ψ}, that

$$|\,\Psi\,(L\chi_{[0,x]}, x)\,| \leqq L(x)\left(\frac{P_{\eta}(x)}{x^{\eta}\,L(x)}\right) x^{\eta}\,W_{\Psi}\,(\chi_{[0,1]}(t) + t^{-\eta}\,\chi_{(1,\infty)}(t), x)\,.$$

By (4.5.1) and hypothesis (3.10), it follows that

$$(4.5.5) \qquad |\,\Psi\,(L\chi_{[0,x]}, x)\,| = O\,(L(x)) \quad (x \to \infty)\,.$$

In a similar way we have

$$L(t)\,\chi_{(x,\infty)}(t) \leqq Q_{\eta}\,(x) t^{\eta}\,,$$

for all $t \geqq 0$, and

$$L(t) = o\,(t^{\eta}) \quad (t \to \infty)\,.$$

Hence, by definition of W_{Ψ}, it follows that

$$|\,\Psi\,(L\chi_{(x,\infty)}, x)\,| \leqq L(x)\left(\frac{Q\,(x)}{x^{-\eta}\;L\,(x)}\right) x^{-\eta}\;W_{\Psi}\,(t^{\eta}, x)\,.$$

Using (4.5.2) and hypothesis (3.9) we find that

$$(4.5.6) \qquad |\,\Psi\,(L\chi_{(x,\infty)}, x)\,| = O\,(L(x)) \quad (x \to \infty)\,.$$

From (4.5.4), (4.5.5) and (4.5.6) follows finally that

$$\Psi\,(L, x) = O\,(L(x)) \quad (x \to \infty)\,.$$

(Necessity). We shall prove first that, if (3.8) is true for all SV functions $L \in \mathcal{M}_0$, then

$$(4.5.7) \qquad W_{\Psi}\,(L, x) = O\,(L(x)) \quad (x \to \infty)\,.$$

Let f be a function in $\mathcal{M}_0$ such that $f\,(x) \to 0$ $(x \to \infty)$, and let

$$l\,(x) = (2\,\|f\| + f\,(x))\,L(x)\,.$$

The function l is clearly a SV function in $\mathcal{M}_0$ and we have

$$\Psi(l, x) = 2\|f\| \Psi(L, x) + \Psi(fL, x).$$

If we define Ψ_L by

$$\Psi_L(f, x) = \frac{1}{L(x)} \Psi(fL, x)$$

then Ψ_L is a regular operator and

$$\text{(4.5.8)} \quad \Psi_L(f, x) = (2\|f\| + f(x)) \frac{1}{l(x)} \Psi(l, x) - \frac{2\|f\|}{L(x)} \Psi(L, x).$$

Since, by hypothesis, $\Psi(l, x) = O(l(x))$ and $\Psi(L, x) = O(L(x))$ $(x \to \infty)$, the operator Ψ_L transforms every function f in $\mathcal{M}_0$ that converges to zero as $x \to \infty$ into a bounded function. Hence by Theorem 2, we must have

$$W_{\Psi_L}(1, x) = O(1) \quad (x \to \infty).$$

Take now any $g \in \mathcal{M}_0$ such that $|g| \leq L$ and $g = o(L)$.

We then have

$$|\Psi(g, x)| = L(x) |\Psi_L(\frac{g}{L}, x)| \leq L(x) W_{\Psi_L}(1, x)$$

and it follows that

$$W_\Psi(L, x) \leq L(x) W_{\Psi_L}(1, x) = O(L(x)) \quad (x \to \infty).$$

Thus (4.5.7) is proved.

Note that we have in particular

$$\text{(4.5.9)} \qquad W_\Psi(1, x) = O(1) \quad (x \to \infty).$$

We shall now prove that relation (4.5.7) implies (3.9).

Suppose by way of contradiction that there exists no $\eta > 0$ such that (3.9) holds. Then

$$\limsup_{x \to \infty} x^{-1/n} W_\Psi(t^{1/n}, x) = \infty, \text{ for } n = 1, 2, \ldots$$

It is then possible to construct by induction a sequence of numbers (x_n) and a sequence (g_n) of functions in $\mathcal{M}_0$ such that for all $n = 1, 2, \ldots,$

$$x_{n+1} \geq 2x_n, \quad x_1 > 0,$$

$$\text{(4.5.10)} \qquad W_\Psi(t^{1/n}, x_n) \geq n x_n^{1/n},$$

$$| g_n(x) | \leq x^{1/n}, \; g_n(x) = o(x^{1/n}) \quad (x \to \infty),$$

(4.5.11) $$| \Psi(g_n, x_n) | \geq \frac{3}{4} W_\Psi(t^{1/n}, x_n)$$

and

(4.5.12) $$| g_n(t) | \leq \frac{1}{2} t^{1/n}, \text{ for } t \geq x_{n+1} .$$

Let

$$\varepsilon(u) = \begin{cases} 0, 0 \leq u < x_1 , \\ \frac{1}{n}, x_n \leq u < x_{n+1}, \; n = 1, 2, ..., \end{cases}$$

and

$$L(x) = \exp\left(\int_0^x \frac{\varepsilon(u)}{u} du\right).$$

L is clearly a continuous and increasing SV function. We shall show that L does not satisfy condition (4.5.7).

If $x_n \leq t < x_{n+1}$, we have

$$\frac{L(t)}{L(x_n)} = \exp\left(\int_{x_n}^t \frac{\varepsilon(u)}{u} du\right) = \left(\frac{t}{x_n}\right)^{1/n}.$$

Since $| g_n(t) | \leq t^{1/n}$ for all $t \in R^+$, we have

$$| g_n(t) | \chi_{[x_n, x_{n+1}]}(t) \leq t^{1/n} \chi_{[x_n, x_{n+1}]}(t)$$
$$\leq x_n^{1/n} \frac{L(t)}{L(x_n)} \chi_{[x_n, x_{n+1}]}(t) \leq x_n^{1/n} \cdot \frac{L(t)}{L(x_n)} .$$

On the other hand

$$| g_n(t) | \chi_{[x_n, x_{n+1}]}(t) = o\left(\frac{L(t)}{L(x_n)} x_n^{1/n}\right) \quad (t \to \infty).$$

Hence, by definition of W_Ψ, for $n = 1, 2, ...$, we have the inequality

(4.5.13) $$| \Psi(g_n \chi_{[x_n, x_{n+1}]}, x_n) | \leq \frac{1}{L(x_n)} x_n^{1/n} W_\Psi(L, x_n) .$$

By linearity of Ψ, we have

$$| \Psi (g_n \chi_{[x_n, x_n+1]}, x) |$$
$$\geqq | \Psi (g_n, x_n) | - | \Psi (g_n \chi_{[0,x_n)}, x_n) | - | \Psi (g_n \chi_{(x_n+1, \infty)}, x_n) | .$$

Using (4.5.11), (4.5.12) and the definition of W_Ψ, we find that

$$(4.5.14) \qquad | \Psi (g_n \chi_{[x_n, x_n+1]}, x_n) |$$
$$\geqq \frac{3}{4} W_\Psi (t^{1/n}, x_n) - x_n^{1/n} W_\Psi (1, x_n) - \frac{1}{2} W_\Psi (t^{1/n}, x_n) .$$

From (4.5.13), (4.5.14) and (4.5.10) it follows that

$$\frac{1}{L(x_n)} W_\Psi (L, x_n) \geqq \frac{1}{4} x_n^{-1/n} W_\Psi (t^{1/n}, x_n) - W_\Psi (1, x_n)$$
$$\geqq \frac{1}{4} n - W_\Psi (1, x_n) \to \infty \quad (n \to \infty) .$$

But this is impossible, by (4.5.7). This contradiction proves the necessity of condition (3.9.)

In order to prove (3.10), observe first that, in view of the inequality

$$W_\Psi (\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,\infty)}(t), x)$$
$$\leqq W_\Psi (\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,x)}(t), x) + x^{-\eta} W_\Psi (1, x) ,$$

which is valid for all $x > 1$, and (4.5.9), it is sufficient to prove that for some $\eta > 0$

$$(4.5.15) \qquad W_\Psi (\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,x)}(t), x) = O (x^{-\eta}) \quad (x \to \infty) .$$

Suppose, by way of contradiction, that (4.5.15) is not true. Let

$$h_n (t) = \chi_{[0,1]}(t) + t^{-1/n} \chi_{(1,\infty)}(t) .$$

Then we have

$$\limsup_{x \to \infty} x^{1/n} W_\Psi (h_n \chi_{[0,x]}, x) = \infty, \quad n = 1, 2, \dots .$$

It follows that we can find a sequence (x_n) of numbers and a sequence (f_n) of functions in $\mathcal{M}_0$ such that

$$x_1 > 1, \ x_n \to \infty \quad (n \to \infty) ,$$

$$(4.5.16) \qquad x_n^{1/n} W_\Psi (h_n \chi_{[0,x_n)}, x_n) \geqq n, \quad n = 1, 2, \dots,$$

$$(4.5.17) \qquad |f_n| \leqq h_n \chi_{[0,x_n]}, \ f_n (t) = o (t^{-1/n}) \quad (t \to \infty)$$

and

$$(4.5.18) \qquad |\Psi(f_n, x_n)| \geqq \frac{3}{4} W_\Psi(h_n \chi_{[0,x_n]}, x_n).$$

Define

$$\varepsilon(u) = \begin{cases} 0, & 0 \leqq u < 1, \\ \frac{1}{n}, & x_{n-1} \leqq u < x_n, \; n = 1, 2, \ldots, \end{cases}$$

where $x_0 = 1$, and let

$$L(x) = \exp\left(-\int_0^x \frac{\varepsilon(u)}{u}\, du\right).$$

The function L is clearly a decreasing and continuous SV function. Moreover, we have

$$(4.5.19) \qquad \frac{L(t)}{L(x_n)} = \left(\frac{t}{x_n}\right)^{-1/n}, \text{ for } x_{n-1} \leqq t < x_n, \quad n = 1, 2, \ldots,$$

and

$$(4.5.20) \quad h_n(t)\, x_n^{-1/n} L(x_n) \leqq L(t), \text{ for } 0 \leqq t \leqq x_n, \quad n = 1, 2, \ldots$$

The first equality follows immediately from the definition of L. As far as (4.5.20) is concerned, for $0 \leqq t < 1$, both sides are equal to 1; for $1 \leqq t \leqq x$, the inequality follows from (4.5.19) by induction: supposing that (4.5.20) is true for some $n = r$, we shall prove that it is true for $n = r + 1$. If $1 \leqq t \leqq x_r$, we have

$$h_{r+1}(t)\, x_{r+1}^{1/r+1} L(x_{r+1}) = \left(\frac{t}{x_{r+1}}\right)^{-1/r+1} L(x_{r+1})$$

$$= \left(\frac{t}{x_r}\right)^{-1/r} L(x_r) \left(\frac{t}{x_r}\right)^{1/r(r+1)} \frac{L(x_r)}{L(x_{r+1})} \left(\frac{x_{r+1}}{x_r}\right)^{1/r+1} \leqq L(t).$$

If $x_r < t \leqq x_{r+1}$, we have by (4.5.19)

$$h_{r+1}(t)\, x_{r+1}^{1/r+1} L(x_{r+1}) = \left(\frac{t}{x_{r+1}}\right)^{-1/r+1} L(x_{r+1}) = L(t).$$

Thus (4.5.20) is proved.

From (4.5.17) and (4.5.18) follows that

$$x_n^{1/n} |f_n(t)| \leqq x_n^{1/n} h_n(t)\, \chi_{[0,x_n]}(t) \leqq \frac{L(t)}{L(x_n)}$$

for all $t \geqq 0$ and

$$x_n^{1/n} f_n(t) = o\left(\frac{L(t)}{L(x_n)}\right) \quad (t \to \infty)$$

since $f_n(t) = 0$ for $t \geqq x_n$. Hence by definition of W_Ψ, (4.5.18) and (4.5.16), we find that

$$\frac{1}{L(x)} W_\Psi(L, x_n) \geqq x_n^{1/n} \,|\, \Psi(f_n, x_n) \,|\, \geqq \frac{3}{4} x_n^{1/n} W_\Psi(h_n \chi_{[0,x_n]}, x_n)$$

$$\geqq \frac{3}{4} n \to \infty \quad (n \to \infty).$$

But this is impossible by (4.5.7). This contradiction proves the necessity of condition (3.10).

4.6. Proof of Theorem 6. (Sufficiency). We have to show that for every SV function $L \in \mathcal{M}_0$

$$\lim_{x \to \infty} \frac{\Psi(L, x)}{L(x)} = 1. \tag{4.6.1}$$

First we have

$$\left| \frac{\Psi(L, x)}{L(x)} - 1 \right| \leqq \left| \Psi\left(\frac{L(t)}{L(x)} - 1, x\right) \right| + |\, \Psi(1, x) - 1 \,|. \tag{4.6.2}$$

Let $0 < \alpha < 1 < \beta < \infty$. Then we have

$$\left| \Psi\left(\frac{L(t)}{L(x)} - 1, x\right) \right|$$

$$\leqq \left| \Psi\left(\left(\frac{L(t)}{L(x)} - 1\right) \chi_{[0,\alpha x)}(t), x\right) \right| + \left| \Psi\left(\left(\frac{L(t)}{L(x)} - 1\right) \chi_{[\alpha x, \beta x]}(t), x\right) \right|$$

$$+ \left| \Psi\left(\left(\frac{L(t)}{L(x)} - 1\right) \chi_{(\beta x, \infty)}(t), x\right) \right| \tag{4.6.3}$$

$$\leqq |\, \Psi_{[0,\alpha x)} \,| + |\, \Psi_{[\alpha x, \beta x]} \,| + |\, \Psi_{(\beta x, \infty)} \,|.$$

As in the proof of Theorem 5, we can show that

$$\left| \frac{L(t)}{L(x)} - 1 \right| \chi_{[0,\alpha x)}(t) \leqq \left(\frac{P_\eta(\alpha x)}{L(x)} + (\alpha x)^\eta \right) \left(\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,\infty)}(t) \right)$$

for $x > 1/\alpha$ and $t \in R^+$. Since the left-hand side of this inequality is zero for $t \geqq x$, we have, by definition of W_Ψ,

$$|\Psi_{[0,\alpha x)}|$$
$$\leqq \left(\frac{P_\eta(\alpha x)}{(\alpha x)^\eta L(\alpha x)} \cdot \frac{L(\alpha x)}{L(x)} + 1 \right) \alpha^\eta x^\eta W_\Psi \left(\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,\infty)}(t),\ x \right).$$

By (4.5.1) and hypothesis (3.10), it follows that

$$\limsup_{x\to\infty} |\Psi_{[0,\alpha x)}| \geqq \alpha^\eta M. \tag{4.6.4}$$

Likewise, for $x > 1/\alpha$ and $t \in R^+$, we have

$$\left| \frac{L(t)}{L(x)} - 1 \right| \chi_{(\beta x,\infty)}(t) \leqq \left(\frac{Q_\eta(\beta x)}{L(x)} + (\beta x)^{-\eta} \right) t^\eta .$$

Since $t^{-\eta} L(t) \to 0$ $(t \to \infty)$, it follows, by definition of W_Ψ, that

$$|\Psi_{(\beta x,\infty)}| \leqq \left(\frac{Q_\eta(\beta x)}{(\beta x)^{-\eta} L(\beta x)} \cdot \frac{L(\beta x)}{L(x)} + 1 \right) \beta^{-\eta} x^{-\eta} W_\Psi(t^\eta, x).$$

By (4.5.2) and hypothesis (3.9) we find that

$$\limsup_{x\to\infty} |\Psi_{(\beta x,\infty)}| \leqq M\beta^{-\eta}. \tag{4.6.5}$$

As for the second term of (4.6.3), we have

$$|\Psi_{[\alpha x,\beta x]}| \leqq \sup_{\alpha x \leqq t \leqq \beta x} \left| \frac{L(t)}{L(x)} - 1 \right| W_\Psi(1, x).$$

From the Representation Theorem for SV functions follows immediately that

$$\sup_{\alpha x \leqq t \leqq \beta x} \left| \frac{L(t)}{L(x)} - 1 \right| = \sup_{\alpha \leqq \lambda \leqq \beta} \left| \frac{L(\lambda x)}{L(x)} - 1 \right| \to 0 \quad (x \to \infty).$$

Hence

$$\lim_{x\to\infty} |\Psi_{[\alpha x,\beta x]}| = 0. \tag{4.6.6}$$

From (4.6.3), (4.6.4), (4.6.5) and (4.6.6) it follows that

$$\limsup_{x\to\infty} \left| \Psi\left(\frac{L(t)}{L(x)} - 1, x \right) \right| \leqq (\alpha^\eta + \beta^{-\eta}) M,$$

and (4.6.1) is proved by choosing α arbitrarily small and β arbitrarily large.

(Necessity). The necessity of (3.12) is obvious. As for (3.8) and (3.9), in view of the proof of Theorem 5, it will be sufficient to show that our hypothesis (3.11) implies (4.5.7).

Let $f \in \mathcal{M}_0$ be such that $\lim_{x \to \infty} f(x) = c$. If L is any SV function in $\mathcal{M}$, let

$$l(x) = (2\|f\| + f(x))\, L(x)\,.$$

The function l is clearly a SV function in $\mathcal{M}_0$ and we have

$$\Psi(fL, x) = \Psi(l, x) - 2\|f\|\, \Psi(L, x)\,.$$

If we define the operator Ψ_L by

$$\Psi_L(f, x) = \frac{1}{L(x)}\, \Psi(Lf, x)\,,$$

then Ψ_L is a regular operator and

$$\Psi_L(f, x) = \frac{1}{L(x)}\, \Psi(fL, x)$$

$$= (2\|f\| + f(x))\, \frac{\Psi(l, x)}{l(x)} - 2\|f\|\, \frac{\Psi(L, x)}{L(x)}\,.$$

By (3.11) we have $\Psi(l, x)/l(x) \to 1$ and $\Psi(L, x)/L(x) \to 1$ $(x \to \infty)$ and so

$$\Psi_L(f, x) \to 2\|f\| + c - 2\|f\| = c \quad (x \to \infty)\,.$$

Hence, by Theorem 3, the operator Ψ_L preserves convergence and consequently

$$W_{\Psi_L}(1, x) = O(1) \quad (x \to \infty)\,.$$

But

$$W_{\Psi_L}(1, x) = \frac{1}{L(x)}\, W_{\Psi}(L, x)$$

and the necessity of (4.5.7) is proved.

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