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ASYMPTOTIC PROPERTIES OF LINEAR OPERATORS [1]

by R. Bojanic [2] and M. Vuilleumier

1. Positive and regular linear operators.

1.1. One of the most interesting and recent developments in the theory of approximation is a systematic and very successful study of approximation properties of sequences of positive linear operators. One of the best known results in this direction is probably the theorem of Korovkin [25] which states that if a sequence of positive linear operators approximates 1, x and x^2 on $[a, b]$, then it approximates every continuous function on $[a, b]$ (see also [26], pp. 192-196). As we shall see below, this result is typical in a certain sense for positive linear operators. Generally speaking, if positive linear operators have a certain property on a small class of functions, in many cases it can be proved that they have the same property on a larger class of functions. One of the principal aims of this paper is to extend the class of positive linear operators to linear operators which are not necessarily positive but preserve this typical property of positive linear operators.

In classical analysis and especially in the theory of summability there are many examples of positive linear operators, such as the Laplace transform $\mathscr{L}(f, .)$ of an integrable function f on $R^+ = \{x : x \geqq 0\}$, defined by

$$\mathscr{L}(f, x) = \frac{1}{x} \int_0^\infty e^{-(t/x)} f(t)\, dt \qquad (x > 0) .$$

One of the basic problems here is to study how well the transform of a function preserves its asymptotic properties. Simplest results of this type for the Laplace transform are:

If f is a bounded, measurable function on R^+, then the Laplace trans-

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form $\mathscr{L}(f, .)$ of f is also a bounded function. If, in addition, $\lim_{x \to \infty} f(x) = \alpha$ then $\lim_{x \to \infty} \mathscr{L}(f, x) = \alpha$.

Results of this type can be easily extended to arbitrary positive linear operators since positive linear operators preserve the inequalities and most of the techniques in asymptotic analysis are based on order properties of R. The results thus obtained are in many respects similar to the theorem of Korovkin. To illustrate this point, let us consider the linear space $\mathscr{F}$ of all real-valued functions on R^+ with the usual order relation $\leqslant$ [1]). Let $\mathscr{F}_0$ be the linear subspace and sublattice of $\mathscr{F}$ consisting of all real-valued functions in $\mathscr{F}$ which are bounded on every finite subinterval of R^+.

We then have the following simple results:

(i) *A positive linear operator $\Phi: \mathscr{F}_0 \to \mathscr{F}$ transforms a bounded function into a bounded function if and only if $\Phi(1, .)$ is a bounded function on R^+.*

This result follows immediately from the inequality

$$|\Phi(f, x)| \leqq \Phi(|f|, x) \leqq \Phi(1, x) \, \|f\|$$

where $\|f\| = \sup\{|f(t)| : t \in R^+\}$.

(ii) *A positive linear operator $\Phi: \mathscr{F}_0 \to \mathscr{F}$ is convergence preserving, i.e.*

$$f \in \mathscr{F}_0 \text{ and } \lim_{t \to \infty} f(t) = c \Rightarrow \lim_{x \to \infty} \Phi(f, x) = c,$$

if $\Phi(1, x) \to 1$ $(x \to \infty)$ and if there is a positive decreasing function $g \in \mathscr{F}_0$ converging to zero as $x \to \infty$ such that $\Phi(g, x) \to 0$ $(x \to \infty)$.

The proof of this result is also very simple. Suppose that $f \in \mathscr{F}_0$ and that $\lim_{t \to \infty} f(t) = c$. Since $\Phi(f, x) - c = \Phi(f - c, x) + c(\Phi(1, x) - 1)$, we have, by positivity of Φ,

$$|\Phi(f, x) - c| \leqq \Phi(|f - c|, x) + |c| \, |\Phi(1, x) - 1|.$$

In order to estimate the first term on the right-hand side of this inequality, observe that for all $t \in R^+$ we have

$$\begin{aligned} |f(t) - c| &\leqq (\|f\| + |c|)\, \chi_{[0,\Delta]}(t) + \sup_{t \geqq \Delta} |f(t) - c| \\ &\leqq \frac{\|f\| + |c|}{g(\Delta)}\, g(t) + \sup_{t \geqq \Delta} |f(t) - c|. \end{aligned}$$

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[1]) $f \leqslant g$ means $f(x) \leqslant g(x)$ for every $x \in R^+$.

Hence

$$\Phi(|f-c|,x) \leqq \frac{\|f\|+|c|}{g(\Delta)} \Phi(g,x) + \sup_{t \geqq \Delta} |f(t)-c| \, \Phi(1,x),$$

and so

$$|\Phi(f,x)-c| \leqq \frac{\|f\|+|c|}{g(\Delta)} \Phi(g,x) + \sup_{t \geqq \Delta} |f(t)-c| \, \Phi(1,x)$$
$$+ |c| \, . \, |\Phi(1,x)-1| \, .$$

Since, by hypotheses, $\Phi(1, x) \to 1$ and $\Phi(g, x) \to 0$ $(x \to \infty)$, it follows that

$$\limsup_{x \to \infty} |\Phi(f,x)-c| \leqq \sup_{t \geqq \Delta} |f(t)-c| \, ,$$

and the result is proved since Δ can be chosen arbitrarily large.

1.2. The most general linear transformations in the theory of summability are not necessarily positive. If $\psi(x, .)$ is a Lebesgue integrable function on R^+ for every fixed $x \in R^+$, and if $\mathcal{S}$ is the family of all measurable functions f on R^+ such that

$$\int_0^\infty |\psi(x,t)| \; |f(t)| \, dt < \infty \text{ for every } x \in R^+ \, ,$$

then $\mathcal{S}$ is a linear space and we can define a linear operator G on $\mathcal{S}$ by

$$G(f,x) = \int_0^\infty \psi(x,t) f(t) \, dt \, . \tag{1.1}$$

We shall consider here, in particular, the subspace $\mathcal{S}_0$ of $\mathcal{S}$, consisting of all functions in $\mathcal{F}$ which are bounded on finite subintervals of R^+.

The classical results of H. Hahn [1] and H. Raff [2], [3] give necessary and sufficient conditions for the operator G to transform every bounded function in $\mathcal{S}$ into an eventually bounded function and a convergent function in $\mathcal{S}_0$ into a convergent function:

A. *In order that, as $x \to \infty$,*

$$f \in \mathcal{S}_0 \text{ and } f(x) = O(1) \quad \Rightarrow \quad G(f,x) = O(1)$$

it is necessary and sufficient that

$$\int_0^\infty |\psi(x,t)| \, dt = O(1) \, .$$

B. In order that

$$f \in \mathcal{S}_0 \text{ and } \lim_{x\to\infty} f(x) = c \quad \Rightarrow \quad \lim_{x\to\infty} G(f, x) = c$$

it is necessary and sufficient that

(i) $\int_0^\infty \psi(x, t)\, dt \to 1 \quad (x \to \infty)$,

(ii) $\int_0^\infty \psi(x, t)\, \chi_E(t)\, dt \to 0 \quad (x \to \infty)$

for all bounded measurable sets $E \subset R^+$,

(iii) $\int_0^\infty |\psi(x, t)|\, dt = O(1) \quad (x \to \infty)$.

In order to extend the preceding results to more general linear operators, let us observe first that the operator (1.1) can be expressed as a difference of two positive linear operators:

$$G(f, x) = \int_0^\infty \psi^+(x, t) f(t)\, dt - \int_0^\infty \psi^-(x, t) f(t)\, dt, \tag{1.2}$$

where, as usual, $a^+ = \max(a, 0)$, $a^- = -\min(a, 0)$.

It seems therefore that the most natural generalization of the operator (1.1) as well as arbitrary positive linear operators, to operators for which the results of type *A* and *B* would be true, should be the class of linear operators which can be expressed as a difference of two positive linear operators. A linear operator which has this property is called a *regular operator*. General theory of regular operators on partially ordered linear spaces can be found in L. V. Kantorovič, B. Z. Vulih, A. G. Pinsker ([5], Ch. VII), B. Z. Vulih ([6], Ch. VIII) and in H. Nakano [7].

For completeness sake, we shall give here an outline of the most important properties of regular operators which will be needed in this paper.

1.3. We shall use here the following definition of regular operators:

Definition: Let $\mathcal{F}_i$, $i = 1, 2$ *be linear subspaces and sublattices of* $\mathcal{F}$. *A linear operator* $\Psi: \mathcal{F}_1 \to \mathcal{F}_2$ *is called a regular operator if there exist positive linear operators* $\Phi_i: \mathcal{F}_1 \to \mathcal{F}$, $i = 1, 2$ *such that* $\Psi(f, x) = \Phi_1(f, x) - \Phi_2(f, x)$ *for every* $f \in \mathcal{F}_1$, *and* $x \in R^+$.

This definition of regular operator does not require the operators Φ_1 and Φ_2 to have values in $\mathscr{F}_2$. In the problems considered here such a requirement is not essential.

We have now the following result.

THEOREM I. *A linear operator* $\Psi: \mathscr{F}_1 \to \mathscr{F}_2$ *is regular if and only if there exists a positive linear operator* $\Phi: \mathscr{F}_1 \to \mathscr{F}$ *such that*

$$|\Psi(f, x)| \leqq \Phi(|f|, x) \tag{1.3}$$

for every $f \in \mathscr{F}_1$ *and* $x \in R^+$.

If Ψ is a regular operator, we have clearly

$$|\Psi(f, x)| \leqq |\Phi_1(f, x)| + |\Phi_2(f, x)|$$
$$\leqq \Phi_1(|f|, x) + \Phi_2(|f|, x) = \Phi(|f|, x)$$

where $\Phi(f, x) = \Phi_1(f, x) + \Phi_2(f, x)$ is a positive linear operator from $\mathscr{F}_1$ into $\mathscr{F}$. Conversely, if (1.3) holds true, let

$$\Phi_1(f, x) = \Phi(f, x),$$
$$\Phi_2(f, x) = \Phi(f, x) - \Psi(f, x).$$

Then $\Psi(f, x) = \Phi_1(f, x) - \Phi_2(f, x)$. Here Φ_1 is obviously a positive linear operator from $\mathscr{F}_1$ into $\mathscr{F}$, and, if $f \geqq 0$ on R^+, we have

$$\Phi_2(f, x) = \Phi(f, x) - \Psi(f, x) \geqq \Phi(f, x) - |\Psi(f, x)| \geqq 0$$

so that Φ_2 is also a positive linear operator from $\mathscr{F}_1$ into $\mathscr{F}$.

In most applications, if a regular operator Ψ is given, it is important to have an intrinsic definition of the operator Φ for which (1.3) is true. This is possible, in view of the following result:

THEOREM II. *A linear operator* $\Psi: \mathscr{F}_1 \to \mathscr{F}_2$ *is regular if and only if for every* $f \in \mathscr{F}_1, f \geqq 0$, *and every* $x \in R^+$ *we have*

$$V_\Psi(f, x) < \infty \tag{1.4}$$

where

$$V_\Psi(f, x) = \sup\{|\Psi(g, x)|: g \in \mathscr{F}_1 \text{ and } |g| \leqq f\}. \tag{1.5}$$

If condition (1.4) is satisfied, we have clearly $V_\Psi(f, .) \in \mathscr{F}$ and the proof that Ψ is regular is carried out in three steps. First, one shows that

V_Ψ is additive on non-negative functions in $\mathscr{F}_1$ and homogeneous with respect to multiplication by positive real numbers:

$$f_1 \geqq 0 \text{ and } f_2 \geqq 0 \Rightarrow V_\Psi(f_1+f_2, x) = V_\Psi(f_1, x) + V_\Psi(f_2, x),$$
$$\alpha > 0 \text{ and } f \geqq 0 \Rightarrow V_\Psi(\alpha f, x) = \alpha V_\Psi(f, x).$$

The operator V_Ψ is then extended to a positive linear operator $\Phi_\Psi : \mathscr{F}_1 \to \mathscr{F}$, which coincides with V_Ψ on non-negative functions, in the usual way: if $f = f^+ - f^-$, then $\Phi_\Psi(f, x) = V_\Psi(f^+, x) - V_\Psi(f^-, x)$. Finally, if $f \in \mathscr{F}_1$ we have $|\Psi(f, x)| \leqq V_\Psi(|f|, x) = \Phi_\Psi(|f|, x)$. Hence, by Theorem I, the operator Ψ is regular.

Conversely, if Ψ is regular, by Theorem I, we have a positive linear operator $\Phi : \mathscr{F}_1 \to \mathscr{F}$ such that

$$|\Psi(g, x)| \leqq \Phi(|g|, x) \leqq \Phi(f, x)$$

for every $g \in \mathscr{F}_1$, $|g| \leqq f$, and every $x \in R^+$. Hence, the condition (1.4) of Theorem II is satisfied.

If we consider the operator G defined by (1.1), then

$$V_G(f, x) = \int_0^\infty |\psi(x, t)| f(t)\, dt .$$

From the statement of the theorems of Hahn and Raff mentioned earlier, we can expect that the operator V_Ψ will play an important role in the extension of these results to general regular operators. In fact, as in the theory of positive linear operators, some asymptotic property of the regular operator Ψ will hold for a large class of functions if and only if the operator V_Ψ has certain properties on a much smaller class of functions.

2. Boundedness and convergence preserving regular operators.

In this section and the following one we shall extend to regular operators some of the well-known results about the asymptotic behavior of the special transform G defined by (1.1).

Let us consider the linear space $\mathscr{M}$ of real valued measurable functions on R^+ and let $\mathscr{M}_0$ be the subspace of $\mathscr{M}$ consisting of all measurable functions on R^+ which are bounded on every finite interval of R^+.

The basic result which characterizes regular operators from $\mathscr{M}_0$ into $\mathscr{F}_0$ that preserve boundedness can be stated as follows: