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$$\Psi(\chi_E, x) \to 0 \tag{2.8}$$

for every bounded measurable subset E of R^+, and

$$W_\Psi(1, x) = O(1). \tag{2.9}$$

3. Transformations of O-regular and slowly varying functions by regular operators.

3.1. The class of positive functions which are eventually bounded away from zero and infinity has been extended to the class of O-regular functions defined as follows:

A positive, measurable function l on R^+ is O-regular if

$$\frac{l(\lambda x)}{l(x)} = O(1) \quad (x \to \infty) \tag{3.1}$$

for every $\lambda > 0$.

For example, any function l such that $ax^\alpha \leqq l(x) \leqq Ax^\alpha$, where $\alpha \in R$, clearly satisfies condition (3.1).

The class of O-regular functions and related classes of functions have been studied extensively by V. G. Avakumović [8, 9, 10, 11], J. Karamata [14], N. K. Bari, S. B. Stečkin [15], M. A. Krasnoselskiĭ, T. B. Rutickiĭ [16], W. Matuszewska [17] and others.

The closely related class of slowly varying (SV) functions, introduced by J. Karamata ([12], [13]), generalizes the class of functions converging to a positive limit. A positive, measurable function L defined on R^+ is a slowly varying function if

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1 \tag{3.2}$$

for every $\lambda > 0$.

Clearly, every measurable function on R^+ which converges to a positive limit as $x \to \infty$ is a SV function. Also, functions like

$$\varphi(x) = \begin{cases} 1, 0 \leqq x < e, \\ \log x, x \geqq e, \end{cases} \quad h(x) = \left(2 + \frac{\sin x}{x}\right) \varphi(x),$$

and their iterations are SV functions. More generally, any measurable function g on R^+ such that $\varphi(x) \leqq g(x) \leqq \varphi(x) + \sqrt{\varphi(x)}$ is a SV function.

The most important properties of O-regular and SV functions can be stated as follows:

Representation Theorems: *If l is an O-regular function, there exist $B > 0$ and bounded measurable functions α and β on $[B, \infty]$ such that*

$$(3.3) \qquad l(x) = \exp\left(\alpha(x) + \int_B^x \frac{\beta(t)}{t}\,dt\right) \text{ for } x \geqq B.$$

If L is a SV function, then for some $B > 0$,

$$(3.4) \qquad L(x) = \exp\left(\eta(x) + \int_B^x \frac{\varepsilon(t)}{t}\,dt\right) \text{ for } x \geqq B,$$

where η and ε are bounded measurable functions on $[B, \infty]$ such that $\eta(x) \to c$ and $\varepsilon(x) \to 0$ $(x \to \infty)$.

A proof of these results for continuous O-regular and SV functions can be found in [12], [13], and [14]. These results were subsequently extended to measurable O-regular and SV functions by a number of authors (see [18] for details).

One of the typical and simplest results about the asymptotic behavior of special linear transforms of SV functions is probably the following result of K. Knopp [19]:

If L is a SV function, and if $L \in \mathcal{M}_0$, then

$$\frac{1}{xL(x)} \int_0^\infty e^{-(t/x)} L(t)\,dt \to 1 \quad (x \to \infty).$$

Similar results involving more or less special transformations have been obtained by G. H. Hardy and W. W. Rogosinski [4], S. Aljančić, R. Bojanić, M. Tomić [20], R. Bojanić and J. Karamata [21], and, in slightly different form, by D. Drasin ([22], Th. 6). The most general result of this type, obtained by M. Vuilleumier [23], [24], can be stated as follows:

Let G be defined by (1.1). *In order that*

$$\frac{G(L, x)}{L(x)} \to 1 \quad (x \to \infty)$$

holds for every SV function $L \in \mathcal{M}_0$ it is necessary and sufficient that, as $x \to \infty$,

(i) $\int_0^\infty \Psi(x, t)\,dt \to 1$,

(ii) *there exists* $\eta > 0$ *such that*

$$\int_0^x |\Psi(x,t)|\, t^{-\eta}\, dt = O(x^{-\eta}) \text{ and } \int_x^\infty |\Psi(x,t)|\, t^{\eta} dt = O(x^{\eta}).$$

3.2. Theorem 1 characterizes boundedness preserving operators. A natural extension of that result is the theorem which characterizes regular operators Ψ with the property that $\Psi(l,x) = O(l(x))$ $(x \to \infty)$ holds for every O-regular function $l \in \mathscr{M}_0$. In this direction we have the following result:

THEOREM 4. *Let* $\Psi: \mathscr{M}_0 \to \mathscr{F}_0$ *be a regular operator. In order that*

$$\Psi(l,x) = O(l(x)) \quad (x \to \infty), \tag{3.5}$$

holds for every O-regular function $l \in \mathscr{M}_0$ it is necessary and sufficient that for all $\alpha > 0$, *as* $x \to \infty$,

$$V_\Psi(t^\alpha, x) = O(x^\alpha) \tag{3.6}$$

and

$$V_\Psi(\chi_{[0,1]}(t) + t^{-\alpha}\chi_{(1,\infty)}(t), x) = O(x^{-\alpha}) \tag{3.7}$$

where V_Ψ is defined by (1.5).

Likewise, as an analog of Theorem 2, the following theorem characterizes regular operators which have the property that

$$\Psi(L,x) = O(L(x)) \quad (x \to \infty)$$

holds for every SV function $L \in \mathscr{M}_0$:

THEOREM 5. *Let* $\Psi: \mathscr{M}_0 \to \mathscr{F}_0$ *be a regular operator. In order that*

$$\Psi(L,x) = O(L(x)) \quad (x \to \infty) \tag{3.8}$$

holds for every SV function $L \in \mathscr{M}_0$ it is necessary and sufficient that there exists $\eta > 0$ *such that, as* $x \to \infty$,

$$W_\Psi(t^\eta, x) = O(x^\eta) \tag{3.9}$$

and

$$W_\Psi(\chi_{[0,1]}(t) + t^{-\eta}\chi_{(1,\infty)}(t), x) = O(x^{-\eta}) \tag{3.10}$$

where W_Ψ is defined by (2.5).

Finally, the analog of Theorem 3 can be stated as follows:

THEOREM 6. *Let $\Psi : \mathcal{M}_0 \to \mathcal{F}_0$ be a regular operator. In order that*

$$\frac{\Psi(L, x)}{L(x)} \to 1 \quad (x \to \infty) \tag{3.11}$$

holds for every SV function $L \in \mathcal{M}_0$ it is necessary and sufficient that

$$\Psi(1, x) \to 1 \quad (x \to \infty), \tag{3.12}$$

and that the asymptotic relations (3.9) *and* (3.10) *hold for some* $\eta > 0$.

4. PROOFS.

4.1. Proof of Theorem 1. The sufficiency of condition (2.2) follows from the inequality

$$|\Psi(f, x)| \leq V_\Psi(1, x) \; \|f\| \, .$$

The necessity of (2.2) is proved by way of contradiction. Suppose that (2.2) is not satisfied. Then

$$\limsup_{x \to \infty} V_\Psi(1, x) = \infty \, . \tag{4.1.1}$$

In view of (4.1.1), (2.1) and the properties of Ψ, it is possible to find by induction an increasing sequence (x_k) going to infinity and a sequence (g_k) of functions in $\mathcal{M}_0$ such that, if A_k is defined by $A_k = V_\Psi(1, x_k)$, then

$$A_1 \geq 16 \text{ and } A_k \geq 16 \, A_{k-1}, \quad k = 2, 3, \ldots, \tag{4.1.2}$$

$$A_k \geq 16 \, (\sup_{x \in R^+} | \, \Psi \, (\sum_{i=1}^{k-1} \frac{g_i}{\sqrt{A_i}} \, , x) \, |)^2, \quad k = 2, 3, \ldots, \tag{4.1.3}$$

and

$$|g_k| \leq 1, \; | \, \Psi(g_k, x_k) \, | \geq \frac{3}{4} A_k, \quad k = 1, 2, \ldots \tag{4.1.4}$$

Let

$$g(x) = \sum_{i=1}^{\infty} \frac{g_i(x)}{\sqrt{A_i}} . \tag{4.1.5}$$