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(2.8) $\Psi(\chi_E, x) \to 0$

for every bounded measurable subset E of R^+ , and

(2.9) $W_{\Psi}(1, x) = O(1).$

3. Transformations of O-regular

AND SLOWLY VARYING FUNCTIONS BY REGULAR OPERATORS.

3.1. The class of positive functions which are eventually bounded away from zero and infinity has been extended to the class of O-regular functions defined as follows:

A positive, measurable function l on R^+ is O-regular if

(3.1)
$$\frac{l(\lambda x)}{l(x)} = O(1) \quad (x \to \infty)$$

for every $\lambda > 0$.

For example, any function l such that $ax^{\alpha} \leq l(x) \leq Ax^{\alpha}$, where $\alpha \in R$, clearly satisfies condition (3.1).

The class of *O*-regular functions and related classes of functions have been studied extensively by V. G. Avakumović [8, 9, 10, 11], J. Karamata [14], N. K. Bari, S. B. Stečkin [15], M. A. Krasnoselskiĭ, T. B. Rutickiĭ [16], W. Matuszewska [17] and others.

The closely related class of slowly varying (SV) functions, introduced by J. Karamata ([12], [13]), generalizes the class of functions converging to a positive limit. A positive, measurable function L defined on R^+ is a slowly varying function if

(3.2)
$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1$$

for every $\lambda > 0$.

Clearly, every measurable function on R^+ which converges to a positive limit as $x \to \infty$ is a SV function. Also, functions like

$$\varphi(x) = \begin{cases} 1, 0 \leq x < e, \\ \log x, x \geq e, \end{cases}, \ h(x) = \left(2 + \frac{\sin x}{x}\right) \varphi(x), \end{cases}$$

and their iterations are SV functions. More generally, any measurable function g on R^+ such that $\varphi(x) \leq g(x) \leq \varphi(x) + \sqrt{\varphi(x)}$ is a SV function.

The most important properties of O-regular and SV functions can be stated as follows:

REPRESENTATION THEOREMS: If l is an O-regular function, there exist B > 0 and bounded measurable functions α and β on $[B, \infty]$ such that

(3.3)
$$l(x) = \exp\left(\alpha(x) + \int_{B}^{x} \frac{\beta(t)}{t} dt\right) for \ x \ge B.$$

If L is a SV function, then for some B > 0,

(3.4)
$$L(x) = \exp\left(\eta(x) + \int_{B}^{x} \frac{\varepsilon(t)}{t} dt\right) for \ x \ge B,$$

where η and ε are bounded measurable functions on $[B, \infty]$ such that $\eta(x) \to c$ and $\varepsilon(x) \to 0 \ (x \to \infty)$.

A proof of these results for continuous O-regular and SV functions can be found in [12], [13], and [14]. These results were subsequently extended to measurable O-regular and SV functions by a number of authors (see [18] for details).

One of the typical and simplest results about the asymptotic behavior of special linear transforms of SV functions is probably the following result of K. Knopp [19]:

If L is a SV function, and if $L \in \mathcal{M}_0$, then

$$\frac{1}{xL(x)} \int_{0}^{\infty} e^{-(t/x)} L(t) dt \to 1 \quad (x \to \infty) .$$

Similar results involving more or less special transformations have been obtained by G. H. Hardy and W. W. Rogosinski [4], S. Aljančić, R. Bojanić, M. Tomić [20], R. Bojanić and J. Karamata [21], and, in slightly different form, by D. Drasin ([22], Th. 6). The most general result of this type, obtained by M. Vuilleumier [23], [24], can be stated as follows:

Let G be defined by (1.1). In order that

$$\frac{G(L,x)}{L(x)} \to 1 \quad (x \to \infty)$$

holds for every SV function $L \in \mathcal{M}_0$ it is necessary and sufficient that, as $x \to \infty$,

(i)
$$\int_{0}^{\infty} \Psi(x,t) dt \to 1$$
,

(ii) there exists $\eta > 0$ such that

$$\int_{0}^{x} |\Psi(x,t)| t^{-\eta} dt = O(x^{-\eta}) \text{ and } \int_{x}^{\infty} |\Psi(x,t)| t^{\eta} dt = O(x^{\eta}).$$

3.2. Theorem 1 characterizes boundedness preserving operators. A natural extension of that result is the theorem which characterizes regular operators Ψ with the property that $\Psi(l, x) = O(l(x))(x \to \infty)$ holds for every O-regular function $l \in \mathcal{M}_0$. In this direction we have the following result:

THEOREM 4. Let $\Psi: \mathcal{M}_0 \to \mathcal{F}_0$ be a regular operator. In order that

(3.5)
$$\Psi(l,x) = O(l(x)) \quad (x \to \infty),$$

holds for every O-regular function $l \in \mathcal{M}_0$ it is necessary and sufficient that for all $\alpha > 0$, as $x \to \infty$,

$$(3.6) V_{\Psi}(t^{\alpha}, x) = O(x^{\alpha})$$

and

(3.7)
$$V_{\Psi}(\chi_{[0,1]}(t) + t^{-\alpha}\chi_{(1,\infty)}(t), x) = O(x^{-\alpha})$$

where V_{Ψ} is defined by (1.5).

Likewise, as an analog of Theorem 2, the following theorem characterizes regular operators which have the property that

 $\Psi(L, x) = O(L(x)) \quad (x \to \infty)$

holds for every SV function $L \in \mathcal{M}_0$:

THEOREM 5. Let $\Psi: \mathcal{M}_0 \to \mathcal{F}_0$ be a regular operator. In order that

(3.8)
$$\Psi(L,x) = O(L(x)) \quad (x \to \infty)$$

holds for every SV function $L \in \mathcal{M}_0$ it is necessary and sufficient that there exists $\eta > 0$ such that, as $x \to \infty$,

$$W_{\Psi}(t^{\eta}, x) = O(x^{\eta})$$

and

(3.10)
$$W_{\Psi}(\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,\infty)}(t), x) = O(x^{-\eta})$$

where W_{Ψ} is defined by (2.5).

Finally, the analog of Theorem 3 can be stated as follows:

THEOREM 6. Let $\Psi: \mathcal{M}_0 \to \mathcal{F}_0$ be a regular operator. In order that

(3.11)
$$\frac{\Psi(L,x)}{L(x)} \to 1 \quad (x \to \infty)$$

holds for every SV function $L \in \mathcal{M}_0$ it is necessary and sufficient that

(3.12)
$$\Psi(1, x) \to 1 \quad (x \to \infty),$$

and that the asymptotic relations (3.9) and (3.10) hold for some $\eta > 0$.

4. PROOFS.

4.1. Proof of Theorem 1. The sufficiency of condition (2.2) follows from the inequality

$$|\Psi(f, x)| \leq V_{\Psi}(1, x) \|f\|.$$

The necessity of (2.2) is proved by way of contradiction. Suppose that (2.2) is not satisfied. Then

(4.1.1)
$$\limsup_{x\to\infty} V_{\Psi}(1,x) = \infty.$$

In view of (4.1.1), (2.1) and the properties of Ψ , it is possible to find by induction an increasing sequence (x_k) going to infinity and a sequence (g_k) of functions in \mathcal{M}_0 such that, if A_k is defined by $A_k = V_{\Psi}(1, x_k)$, then

(4.1.2)
$$A_1 \ge 16 \text{ and } A_k \ge 16 A_{k-1}, \quad k = 2, 3, ...,$$

(4.1.3)
$$A_k \ge 16 (\sup_{x \in \mathbb{R}^+} |\Psi| (\sum_{i=1}^{k-1} \frac{g_i}{\sqrt{A_i}}, x) |)^2, \quad k = 2, 3, ...,$$

and

(4.1.4)
$$|g_k| \leq 1, |\Psi(g_k, x_k)| \geq \frac{3}{4} A_k, k = 1, 2, ...$$

Let

(4.1.5)
$$g(x) = \sum_{i=1}^{\infty} \frac{g_i(x)}{\sqrt{A_i}}.$$