Finally, the analog of Theorem 3 can be stated as follows:

THEOREM 6. *Let $\Psi: \mathcal{M}_0 \to \mathcal{F}_0$ be a regular operator. In order that*

$$\frac{\Psi(L,x)}{L(x)} \to 1 \quad (x \to \infty) \tag{3.11}$$

holds for every SV function $L \in \mathcal{M}_0$ it is necessary and sufficient that

$$\Psi(1,x) \to 1 \quad (x \to \infty), \tag{3.12}$$

and that the asymptotic relations (3.9) *and* (3.10) *hold for some* $\eta > 0$.

4. PROOFS.

4.1. Proof of Theorem 1. The sufficiency of condition (2.2) follows from the inequality

$$|\Psi(f,x)| \leqq V_\Psi(1,x) \; \|f\| .$$

The necessity of (2.2) is proved by way of contradiction. Suppose that (2.2) is not satisfied. Then

$$\limsup_{x\to\infty} V_\Psi(1,x) = \infty . \tag{4.1.1}$$

In view of (4.1.1), (2.1) and the properties of Ψ, it is possible to find by induction an increasing sequence (x_k) going to infinity and a sequence (g_k) of functions in $\mathcal{M}_0$ such that, if A_k is defined by $A_k = V_\Psi(1, x_k)$, then

$$A_1 \geqq 16 \text{ and } A_k \geqq 16\, A_{k-1}, \quad k = 2, 3, ..., \tag{4.1.2}$$

$$A_k \geqq 16 \,(\sup_{x\varepsilon R^+} |\, \Psi\,(\sum_{i=1}^{k-1} \frac{g_i}{\sqrt{A_i}}, x)\,|)^2, \quad k = 2, 3, ..., \tag{4.1.3}$$

and

$$|g_k| \leqq 1, \; |\Psi(g_k, x_k)| \geqq \frac{3}{4} A_k, \quad k = 1, 2, ... \tag{4.1.4}$$

Let

$$g(x) = \sum_{i=1}^{\infty} \frac{g_i(x)}{\sqrt{A_i}}. \tag{4.1.5}$$

By (4.1.2) and (4.1.4), this series is uniformly convergent and consequently g is in $\mathscr{M}$. Also, g is bounded on R^+ since

$$|g(x)| \leqq \sum_{i=1}^{\infty} \frac{|g_i(x)|}{4^i} \leqq \frac{1}{3}.$$

We shall show now that

$$|\Psi(g, x)| \to \infty \quad (x \to \infty), \tag{4.1.6}$$

which is impossible by (2.1). Hence, (2.2) must be satisfied.

From the definition of g follows that

$$|\Psi(g, x_k)| \geqq \frac{|\Psi(g_k, x_k)|}{\sqrt{A_k}} - \left|\Psi\left(\sum_{i=1}^{k-1} \frac{g_i}{\sqrt{A_i}}, x_k\right)\right| - \left|\Psi\left(\sum_{i=k+1}^{\infty} \frac{g_i}{\sqrt{A_i}}, x_k\right)\right|.$$

By (4.1.3) we have

$$\left|\Psi\left(\sum_{i=1}^{k-1} \frac{g_i}{\sqrt{A_i}}, x_k\right)\right| \leqq \frac{\sqrt{A_k}}{4}.$$

Finally, by (4.1.4) and (4.1.2)

$$\left|\sum_{i=k+1}^{\infty} \frac{g_i(t)}{\sqrt{A_i}}\right| \leqq \sum_{i=k+1}^{\infty} \frac{1}{\sqrt{A_i}} \leqq \frac{1}{\sqrt{A_k}} \sum_{i=k+1}^{\infty} \frac{1}{4^{i-k}} \leqq \frac{1}{3\sqrt{A_k}}.$$

Since Ψ is a regular operator, it follows that

$$\left|\Psi\left(\sum_{i=k+1}^{\infty} \frac{g_i}{\sqrt{A_i}}, x_k\right)\right| \leqq \frac{1}{3\sqrt{A_k}} V_\Psi(1, x_k) = \frac{1}{3}\sqrt{A_k}.$$

From these inequalities follows that

$$|\Psi(g, x_k)| \geqq \frac{3}{4}\sqrt{A_k} - \frac{1}{4}\sqrt{A_k} - \frac{1}{3}\sqrt{A_k} = \frac{1}{6}\sqrt{A_k} \geqq \frac{1}{6}4^k,$$

and (4.1.6) is proved.

The arguments used here are essentially the same as the ones in the proof of Nakano's Theorem [6, Ch. IX] that the limit of a sequence of regular functionals is a regular functional.

4.2. Proof of Theorem 2. The proof of Theorem 2 is quite similar. The sufficiency of condition (2.4) follows from the inequality

$$|\Psi(f,x)| \leqq W_\Psi(1,x)\,\|f\|\,.$$

The necessity of condition (2.4) is proved by way of contradiction. If (2.4) is not satisfied, it is possible to construct by induction an increasing sequence (x_k) going to infinity and a sequence (g_k) of functions in $\mathcal{M}_0$ such that, if A_k is defined by $A_k = W_\Psi(1, x_k)$, the inequalities (4.1.2), (4.1.3) and (4.1.4) are satisfied and moreover

$$|g(x)| < \frac{1}{2^k}, \text{ for all } x \geqq x_k, \quad k = 2, 3, \ldots$$

and

$$g_k(x) \to 0 \quad (x \to \infty)\,.$$

The function g defined by (4.1.5) has then the properties

$$g(x) \to 0 \quad (x \to \infty)$$

and

$$|\Psi(g, x_k)| \to \infty \quad (k \to \infty)\,.$$

This contradicts hypothesis (2.3) and the necessity of condition (2.4) is proved.

4.3. *Proof of Theorem 3.* (Sufficiency). We have

$$|\Psi(f,x) - c| \leqq |\Psi(f-c,x)| + |c|\,.\,|\Psi(1,x) - 1|\,.$$

Given $\varepsilon > 0$, let X_ε be such that $|f(t) - c| \leqq \varepsilon$ for all $t \geqq X_\varepsilon$ and let

$$g_1(t) = (f(t) - c)\,\chi_{[0,X_\varepsilon]}(t)\,,$$
$$g_2(t) = (f(t) - c)\,\chi_{(X_\varepsilon,\infty)}(t)\,.$$

We then have

$$|\Psi(f - c, x)| \leqq |\Psi(g_1, x)| + |\Psi(g_2, x)|\,.$$

Hence,

$$|\Psi(f,x) - c| \leqq |\Psi(g_1,x)| + |\Psi(g_2,x)| + |c|\ |\Psi(1,x) - 1)|\,. \tag{4.3.1}$$

First, we have $|g_2(t)| \leqq \varepsilon$ for every $t \in R^+$ and $g_2 = o(1)$. Hence, by definition of W_Ψ,

$$|\Psi(g_2,x)| \leqq \varepsilon\, W_\Psi(1,x)\,. \tag{4.3.2}$$

Next, we can find a simple function $h = \sum_{i=1}^{N} A_i \chi_{E_i}$, where E_i, $i = 1,\dots, N$ are measurable subsets of $[0, X_\varepsilon]$, such that

$$|h(t)| \leqq \|g\| \chi_{[0,X_\varepsilon]}(t) \text{ and } \|g - h\| < \varepsilon .$$

Then

$$\begin{aligned}(4.3.3) \qquad |\Psi(g_1, x)| &\leqq |\Psi(g_1 - h, x)| + |\Psi(h, x)| \\ &\leqq \varepsilon W_\Psi(1, x) + \sum_{i=1}^{N} |A_i| \; |\Psi(\chi_{E_i}, x)| .\end{aligned}$$

From (4.3.1), (4.3.2), (4.3.3) and the hypotheses (2.7) and (2.8) follows finally that

$$\limsup_{x \to \infty} |\Psi(f, x) - c| \leqq 2\varepsilon \| W_\Psi(1, .)\|$$

and Theorem 1 is proved since ε can be chosen arbitrarily small.

(Necessity). The necessity of condition (2.9) follows from Theorem 2. The necessity of conditions (2.7) and (2.8) is obvious.

4.4. Proof of Theorem 4. (Sufficiency). Let l be any O-regular function in $\mathscr{M}_0$. Define p_α and q_α by

$$(4.4.1) \qquad p_\alpha(x) = \sup_{0 \leqq t \leqq x} l(t)\left(\chi_{[0,1]}(t) + t^\alpha \chi_{(1,\infty)}(t)\right)$$

and

$$(4.4.2) \qquad q_\alpha(x) = \sup_{t \geqq x} l(t)\, t^{-\alpha} ,$$

Then it can be shown, using representation (3.3), that there exists $\alpha > 0$ such that

$$(4.4.3) \qquad p_\alpha(x) = O\left(x^\alpha l(x)\right) \quad (x \to \infty)$$

and

$$(4.4.4) \qquad q_\alpha(x) = O\left(x^{-\alpha} l(x)\right) \quad (x \to \infty) .$$

To show that (3.5) is satisfied, we start with the inequality

$$(4.4.5) \qquad |\Psi(l, x)| \leqq |\Psi(l\chi_{[0,x]}, x)| + |\Psi(l\chi_{(x,\infty)}, x)| .$$

First we have by (4.4.1)

$$
\begin{aligned}
& l(t)\, \chi_{[0,x]}(t) \\
&= l(t)\big(\chi_{[0,1]}(t) + t^{\alpha}\chi_{(1,\infty)}(t)\big)\, \chi_{[0,x]}(t)\big(\chi_{[0,1]}(t) + t^{-\alpha}\chi_{(1,\infty)}(t)\big) \\
&\leqq p_\alpha(x)\big(\chi_{[0,1]}(t) + t^{-\alpha}\chi_{(1,\infty)}(t)\big)
\end{aligned}
$$

for all $t \geqq 0$. Likewise, by (4.4.2), we have

$$l(t)\, \chi_{(x,\infty)}(t) \leqq q_\alpha(x)\, t^{\alpha}$$

for all $t \geqq 0$. By definition of V_Ψ and (4.4.5), it follows then that

$$|\,\Psi(l, x)\,| \leqq p_\alpha(x)\, V_\Psi\big(\chi_{[0,1]}(t) + t^{-\alpha}\, \chi_{(1,\infty)}(t), x\big) + q_\alpha(x)\, V_\Psi(t^{\alpha}, x)\,.$$

Hence

$$
\begin{aligned}
\frac{1}{l(x)}\,|\,\Psi(l, x)\,| \leqq & \left(\frac{p_\alpha(x)}{x^{\alpha}\, l(x)}\right) x^{\alpha}\, V_\Psi\big(\chi_{[0,1]}(t) + t^{-\alpha}\, \chi_{(1,\infty)}(t), x\big) \\
& + \left(\frac{q_\alpha(x)}{x^{-\alpha}\, l(x)}\right) x^{-\alpha}\, V_\Psi(t^{\alpha}, x)\,,
\end{aligned}
$$

and (3.5) follows from (4.4.3), (4.4.4) and hypotheses (3.6) and (3.7).

(Necessity). Let $\alpha > 0$ and let $f \in \mathscr{M}_0$ be a bounded function on R^{+}. Let

$$g(x) = (2\,\|f\| + f(x))\, x^{\alpha}\,.$$

Then g is an O-regular function, and

$$
\begin{aligned}
\frac{1}{x^{\alpha}}\, \Psi\big(f(t)\, t^{\alpha}, x\big) &= \frac{1}{x^{\alpha}}\, \Psi(g, x) - \frac{2\,\|f\|}{x^{\alpha}}\, \Psi(t^{\alpha}, x) \\
&= (2\,\|f\| + f(x))\, \frac{\Psi(g, x)}{g(x)} - \frac{2\,\|f\|}{x^{\alpha}}\, \Psi(t^{\alpha}, x)\,.
\end{aligned}
$$

Hence, by (3.5), we have

$$\Psi\big(f(t)\, t^{\alpha}, x\big) = O(x^{\alpha}) \quad (x \to \infty)\,,$$

for every bounded function f in $\mathscr{M}_0$. Thus, the regular operator Ψ_α defined by

$$\Psi_\alpha(f, x) = \frac{1}{x^{\alpha}}\, \Psi\big(f(t)\, t^{\alpha}, x\big)$$

transforms every bounded function in $\mathscr{M}_0$ into a bounded function. By Theorem 1, it follows that

$$V_{\Psi_\alpha}(1, x) = O(1) \quad (x \to \infty)\,. \tag{4.4.6}$$

But given any $g \in \mathcal{M}_0$ such that $| g(t) | \leqq t^\alpha$, we have

$$| \Psi(g, x) | = \left| \Psi\left(\frac{g(t)}{t^\alpha} t^\alpha, x\right) \right| = x^\alpha \left| \Psi_\alpha\left(\frac{g(t)}{t^\alpha}, x\right) \right| \leqq x^\alpha V_{\Psi\alpha}(1, x) .$$

Hence, the supremum of the left hand side over all $g \in \mathcal{M}_0$ such that $| g(t) | \leqq t^\alpha$ must satisfy the same inequality:

$$| V_\Psi(t^\alpha, x) | \leqq x^\alpha V_{\Psi\alpha}(1, x)$$

and (3.6) follows by (4.4.6).

The proof of (3.7) is similar to that of (3.6) except that the function t^α, $\alpha > 0$, has to be replaced in the argument by the function $\chi_{[0,1]}(t) + t^{-\alpha} \chi_{(1,\infty)}(t)$.

4.5. *Proof of Theorem 5.* (Sufficiency). Given any *SV* function $L \in \mathcal{M}_0$ and any $\eta > 0$, let

$$P_\eta(x) = \sup_{0 \leqq t \leqq x} t^\eta L(t)$$

and

$$Q_\eta(x) = \sup_{t \geqq x} t^{-\eta} L(t) .$$

Then

$$\frac{P_\eta(x)}{x^\eta L(x)} \to 1 \quad (x \to \infty) \tag{4.5.1}$$

and

$$\frac{Q_\eta(x)}{x^{-\eta} L(x)} \to 1 \quad (x \to \infty) . \tag{4.5.2}$$

The proofs of these relations for continuous *SV* functions can be found in [12] and [13]. For measurable *SV* functions, the proofs follow easily from the representation theorem.

Clearly, if P_η is defined by

$$P_\eta(x) = \sup_{0 \leqq t \leqq x} \left(\chi_{[0,1]}(t) + t^\eta \chi_{(1,\infty)}(t)\right) L(t) , \tag{4.5.3}$$

it will have again the property (4.5.1).

To prove that (3.8) is satisfied, we start with the inequality

$$| \Psi(L, x) | \leqq | \Psi(L\chi_{[0,x]}, x) | + | \Psi(L\chi_{(x,\infty)}, x) | . \tag{4.5.4}$$

First we have by (4.5.3),

$$L(t)\,\chi_{[0,x]}(t)$$
$$= \left(\chi_{[0,1]}(t) + t^{\eta}\chi_{(1,\infty)}(t)\right) L(t)\,\chi_{[0,x]}(t)\left(\chi_{[0,1]}(t) + t^{-\eta}\,\chi_{(1,\infty)}(t)\right)$$
$$\leqq P_{\eta}(x)\left(\chi_{[0,1]}(t) + t^{-\alpha}\,\chi_{(1,\infty)}(t)\right)$$

for all $t \geqq 0$. Since

$$L(t)\,\chi_{[0,x]}(t) = o(t^{-\eta}) \quad (t \to \infty),$$

it follows, by definition of W_{Ψ}, that

$$|\,\Psi(L\chi_{[0,x]}, x)\,| \leqq L(x)\left(\frac{P_{\eta}(x)}{x^{\eta}\,L(x)}\right) x^{\eta}\,W_{\Psi}\left(\chi_{[0,1]}(t) + t^{-\eta}\,\chi_{(1,\infty)}(t), x\right).$$

By (4.5.1) and hypothesis (3.10), it follows that

$$|\,\Psi(L\chi_{[0,x]}, x)\,| = O\left(L(x)\right) \quad (x \to \infty). \tag{4.5.5}$$

In a similar way we have

$$L(t)\,\chi_{(x,\infty)}(t) \leqq Q_{\eta}(x) t^{\eta},$$

for all $t \geqq 0$, and

$$L(t) = o(t^{\eta}) \quad (t \to \infty).$$

Hence, by definition of W_{Ψ}, it follows that

$$|\,\Psi(L\chi_{(x,\infty)}, x)\,| \leqq L(x)\left(\frac{Q(x)}{x^{-\eta}\;L(x)}\right) x^{-\eta}\,W_{\Psi}(t^{\eta}, x).$$

Using (4.5.2) and hypothesis (3.9) we find that

$$|\,\Psi(L\chi_{(x,\infty)}, x)\,| = O\left(L(x)\right) \quad (x \to \infty). \tag{4.5.6}$$

From (4.5.4), (4.5.5) and (4.5.6) follows finally that

$$\Psi(L, x) = O\left(L(x)\right) \quad (x \to \infty).$$

(Necessity). We shall prove first that, if (3.8) is true for all SV functions $L \in \mathcal{M}_0$, then

$$W_{\Psi}(L, x) = O\left(L(x)\right) \quad (x \to \infty). \tag{4.5.7}$$

Let f be a function in $\mathcal{M}_0$ such that $f(x) \to 0$ $(x \to \infty)$, and let

$$l(x) = (2\,\|f\| + f(x))\,L(x).$$

The function l is clearly a SV function in $\mathcal{M}_0$ and we have

$$\Psi(l,x) = 2\|f\|\,\Psi(L,x) + \Psi(fL,x).$$

If we define Ψ_L by

$$\Psi_L(f,x) = \frac{1}{L(x)}\,\Psi(fL,x)$$

then Ψ_L is a regular operator and

$$\Psi_L(f,x) = (2\|f\| + f(x))\frac{1}{l(x)}\,\Psi(l,x) - \frac{2\|f\|}{L(x)}\,\Psi(L,x). \tag{4.5.8}$$

Since, by hypothesis, $\Psi(l,x) = O(l(x))$ and $\Psi(L,x) = O(L(x))$ $(x\to\infty)$, the operator Ψ_L transforms every function f in $\mathcal{M}_0$ that converges to zero as $x\to\infty$ into a bounded function. Hence by Theorem 2, we must have

$$W_{\Psi_L}(1,x) = O(1) \quad (x\to\infty).$$

Take now any $g\in\mathcal{M}_0$ such that $|g|\leqq L$ and $g = o(L)$.

We then have

$$|\Psi(g,x)| = L(x)\,|\Psi_L(\frac{g}{L},x)| \leqq L(x)\,W_{\Psi_L}(1,x)$$

and it follows that

$$W_\Psi(L,x) \leqq L(x)\,W_{\Psi_L}(1,x) = O(L(x)) \quad (x\to\infty).$$

Thus (4.5.7) is proved.

Note that we have in particular

$$W_\Psi(1,x) = O(1) \quad (x\to\infty). \tag{4.5.9}$$

We shall now prove that relation (4.5.7) implies (3.9).

Suppose by way of contradiction that there exists no $\eta > 0$ such that (3.9) holds. Then

$$\limsup_{x\to\infty} x^{-1/n}\,W_\Psi(t^{1/n},x) = \infty\,, \text{ for } n = 1,2,\ldots$$

It is then possible to construct by induction a sequence of numbers (x_n) and a sequence (g_n) of functions in $\mathcal{M}_0$ such that for all $n = 1, 2, \ldots$,

$$x_{n+1} \geqq 2x_n, \quad x_1 > 0,$$

$$W_\Psi(t^{1/n},x_n) \geqq n x_n^{1/n}, \tag{4.5.10}$$

$$| g_n(x) | \leqq x^{1/n}, \; g_n(x) = o(x^{1/n}) \quad (x \to \infty),$$

(4.5.11) $$| \Psi(g_n, x_n) | \geqq \frac{3}{4} W_\Psi(t^{1/n}, x_n)$$

and

(4.5.12) $$| g_n(t) | \leqq \frac{1}{2} t^{1/n}, \text{ for } t \geqq x_{n+1}.$$

Let

$$\varepsilon(u) = \begin{cases} 0, 0 \leqq u < x_1, \\ \frac{1}{n}, x_n \leqq u < x_{n+1}, \; n = 1, 2, \ldots, \end{cases}$$

and

$$L(x) = \exp\left(\int_0^x \frac{\varepsilon(u)}{u} du\right).$$

L is clearly a continuous and increasing SV function. We shall show that L does not satisfy condition (4.5.7).

If $x_n \leqq t < x_{n+1}$, we have

$$\frac{L(t)}{L(x_n)} = \exp\left(\int_{x_n}^t \frac{\varepsilon(u)}{u} du\right) = \left(\frac{t}{x_n}\right)^{1/n}.$$

Since $| g_n(t) | \leqq t^{1/n}$ for all $t \in R^+$, we have

$$| g_n(t) | \chi_{[x_n, x_{n+1}]}(t) \leqq t^{1/n} \chi_{[x_n, x_{n+1}]}(t)$$
$$\leqq x_n^{1/n} \frac{L(t)}{L(x_n)} \chi_{[x_n, x_{n+1}]}(t) \leqq x_n^{1/n} \cdot \frac{L(t)}{L(x_n)}.$$

On the other hand

$$| g_n(t) | \chi_{[x_n, x_{n+1}]}(t) = o\left(\frac{L(t)}{L(x_n)} x_n^{1/n}\right) \quad (t \to \infty).$$

Hence, by definition of W_Ψ, for $n = 1, 2, \ldots$, we have the inequality

(4.5.13) $$| \Psi(g_n \chi_{[x_n, x_{n+1}]}, x_n) | \leqq \frac{1}{L(x_n)} x_n^{1/n} W_\Psi(L, x_n).$$

By linearity of Ψ, we have

$$|\Psi(g_n \chi_{[x_n, x_n+1]}, x)|$$
$$\geqq |\Psi(g_n, x_n)| - |\Psi(g_n \chi_{[0,x_n)}, x_n)| - |\Psi(g_n \chi_{(x_n+1,\infty)}, x_n)|.$$

Using (4.5.11), (4.5.12) and the definition of W_Ψ, we find that

$$|\Psi(g_n \chi_{[x_n, x_n+1]}, x_n)| \tag{4.5.14}$$
$$\geqq \frac{3}{4} W_\Psi(t^{1/n}, x_n) - x_n^{1/n} W_\Psi(1, x_n) - \frac{1}{2} W_\Psi(t^{1/n}, x_n).$$

From (4.5.13), (4.5.14) and (4.5.10) it follows that

$$\frac{1}{L(x_n)} W_\Psi(L, x_n) \geqq \frac{1}{4} x_n^{-1/n} W_\Psi(t^{1/n}, x_n) - W_\Psi(1, x_n)$$
$$\geqq \frac{1}{4} n - W_\Psi(1, x_n) \to \infty \quad (n \to \infty).$$

But this is impossible, by (4.5.7). This contradiction proves the necessity of condition (3.9.)

In order to prove (3.10), observe first that, in view of the inequality

$$W_\Psi(\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,\infty)}(t), x)$$
$$\leqq W_\Psi(\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,x)}(t), x) + x^{-\eta} W_\Psi(1, x),$$

which is valid for all $x > 1$, and (4.5.9), it is sufficient to prove that for some $\eta > 0$

$$W_\Psi(\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,x)}(t), x) = O(x^{-\eta}) \quad (x \to \infty). \tag{4.5.15}$$

Suppose, by way of contradiction, that (4.5.15) is not true. Let

$$h_n(t) = \chi_{[0,1]}(t) + t^{-1/n} \chi_{(1,\infty)}(t).$$

Then we have

$$\limsup_{x \to \infty} x^{1/n} W_\Psi(h_n \chi_{[0,x]}, x) = \infty, \quad n = 1, 2, \ldots.$$

It follows that we can find a sequence (x_n) of numbers and a sequence (f_n) of functions in $\mathcal{M}_0$ such that

$$x_1 > 1, \ x_n \to \infty \quad (n \to \infty),$$

$$x_n^{1/n} W_\Psi(h_n \chi_{[0,x_n)}, x_n) \geqq n, \quad n = 1, 2, \ldots, \tag{4.5.16}$$

$$|f_n| \leqq h_n \chi_{[0,x_n]}, \ f_n(t) = o(t^{-1/n}) \quad (t \to \infty) \tag{4.5.17}$$

and

(4.5.18) $$|\Psi(f_n, x_n)| \geq \frac{3}{4} W_\Psi(h_n \chi_{[0,x_n]}, x_n).$$

Define

$$\varepsilon(u) = \begin{cases} 0, & 0 \leq u < 1, \\ \frac{1}{n}, & x_{n-1} \leq u < x_n, \quad n = 1, 2, \ldots, \end{cases}$$

where $x_0 = 1$, and let

$$L(x) = \exp\left(-\int_0^x \frac{\varepsilon(u)}{u} du\right).$$

The function L is clearly a decreasing and continuous SV function. Moreover, we have

(4.5.19) $$\frac{L(t)}{L(x_n)} = \left(\frac{t}{x_n}\right)^{-1/n}, \text{ for } x_{n-1} \leq t < x_n, \quad n = 1, 2, \ldots,$$

and

(4.5.20) $$h_n(t)\, x_n^{-1/n} L(x_n) \leq L(t), \text{ for } 0 \leq t \leq x_n, \quad n = 1, 2, \ldots$$

The first equality follows immediately from the definition of L. As far as (4.5.20) is concerned, for $0 \leq t < 1$, both sides are equal to 1; for $1 \leq t \leq x$, the inequality follows from (4.5.19) by induction: supposing that (4.5.20) is true for some $n = r$, we shall prove that it is true for $n = r + 1$. If $1 \leq t \leq x_r$, we have

$$h_{r+1}(t)\, x_{r+1}^{1/r+1} L(x_{r+1}) = \left(\frac{t}{x_{r+1}}\right)^{-1/r+1} L(x_{r+1})$$

$$= \left(\frac{t}{x_r}\right)^{-1/r} L(x_r) \left(\frac{t}{x_r}\right)^{1/r(r+1)} \frac{L(x_r)}{L(x_{r+1})} \left(\frac{x_{r+1}}{x_r}\right)^{1/r+1} \leq L(t).$$

If $x_r < t \leq x_{r+1}$, we have by (4.5.19)

$$h_{r+1}(t)\, x_{r+1}^{1/r+1} L(x_{r+1}) = \left(\frac{t}{x_{r+1}}\right)^{-1/r+1} L(x_{r+1}) = L(t).$$

Thus (4.5.20) is proved.

From (4.5.17) and (4.5.18) follows that

$$x_n^{1/n} |f_n(t)| \leq x_n^{1/n} h_n(t)\, \chi_{[0,x_n]}(t) \leq \frac{L(t)}{L(x_n)}$$

for all $t \geqq 0$ and

$$x_n^{1/n} f_n(t) = o\left(\frac{L(t)}{L(x_n)}\right) \quad (t \to \infty)$$

since $f_n(t) = 0$ for $t \geqq x_n$. Hence by definition of W_Ψ, (4.5.18) and (4.5.16), we find that

$$\frac{1}{L(x)} W_\Psi(L, x_n) \geqq x_n^{1/n} \,|\, \Psi(f_n, x_n) \,|\, \geqq \frac{3}{4} x_n^{1/n} W_\Psi(h_n \chi_{[0,x_n]}, x_n)$$

$$\geqq \frac{3}{4} n \to \infty \quad (n \to \infty).$$

But this is impossible by (4.5.7). This contradiction proves the necessity of condition (3.10).

4.6. Proof of Theorem 6. (Sufficiency). We have to show that for every *SV* function $L \in \mathcal{M}_0$

$$\lim_{x \to \infty} \frac{\Psi(L, x)}{L(x)} = 1. \tag{4.6.1}$$

First we have

$$\left| \frac{\Psi(L, x)}{L(x)} - 1 \right| \leqq \left| \Psi\left(\frac{L(t)}{L(x)} - 1, x\right) \right| + |\, \Psi(1, x) - 1 \,|. \tag{4.6.2}$$

Let $0 < \alpha < 1 < \beta < \infty$. Then we have

$$\left| \Psi\left(\frac{L(t)}{L(x)} - 1, x\right) \right|$$

$$\leqq \left| \Psi\left(\left(\frac{L(t)}{L(x)} - 1\right) \chi_{[0,\alpha x)}(t), x\right) \right| + \left| \Psi\left(\left(\frac{L(t)}{L(x)} - 1\right) \chi_{[\alpha x, \beta x]}(t), x\right) \right|$$

$$+ \left| \Psi\left(\left(\frac{L(t)}{L(x)} - 1\right) \chi_{(\beta x, \infty)}(t), x\right) \right| \tag{4.6.3}$$

$$\leqq |\, \Psi_{[0,\alpha x)} \,| + |\, \Psi_{[\alpha x, \beta x]} \,| + |\, \Psi_{(\beta x, \infty)} \,|.$$

As in the proof of Theorem 5, we can show that

$$\left| \frac{L(t)}{L(x)} - 1 \right| \chi_{[0,\alpha x)}(t) \leqq \left(\frac{P_\eta(\alpha x)}{L(x)} + (\alpha x)^\eta \right) \left(\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,\infty)}(t) \right)$$

for $x > 1/\alpha$ and $t \in R^+$. Since the left-hand side of this inequality is zero for $t \geqq x$, we have, by definition of W_Ψ,

$$| \Psi_{[0,\alpha x)} |$$

$$\leqq \left(\frac{P_\eta(\alpha x)}{(\alpha x)^\eta L(\alpha x)} \cdot \frac{L(\alpha x)}{L(x)} + 1 \right) \alpha^\eta x^\eta W_\Psi \left(\chi_{[0,1]}(t) + t^{-\eta} \chi_{(1,\infty)}(t),\ x \right).$$

By (4.5.1) and hypothesis (3.10), it follows that

$$\limsup_{x \to \infty} | \Psi_{[0,\alpha x)} | \geqq \alpha^\eta M . \tag{4.6.4}$$

Likewise, for $x > 1/\alpha$ and $t \in R^+$, we have

$$\left| \frac{L(t)}{L(x)} - 1 \right| \chi_{(\beta x,\infty)}(t) \leqq \left(\frac{Q_\eta(\beta x)}{L(x)} + (\beta x)^{-\eta} \right) t^\eta .$$

Since $t^{-\eta} L(t) \to 0\ (t \to \infty)$, it follows, by definition of W_Ψ, that

$$| \Psi_{(\beta x,\infty)} | \leqq \left(\frac{Q_\eta(\beta x)}{(\beta x)^{-\eta} L(\beta x)} \cdot \frac{L(\beta x)}{L(x)} + 1 \right) \beta^{-\eta} x^{-\eta} W_\Psi(t^\eta, x) .$$

By (4.5.2) and hypothesis (3.9) we find that

$$\limsup_{x \to \infty} | \Psi_{(\beta x,\infty)} | \leqq M \beta^{-\eta} . \tag{4.6.5}$$

As for the second term of (4.6.3), we have

$$| \Psi_{[\alpha x,\beta x]} | \leqq \sup_{\alpha x \leqq t \leqq \beta x} \left| \frac{L(t)}{L(x)} - 1 \right| W_\Psi(1, x) .$$

From the Representation Theorem for *SV* functions follows immediately that

$$\sup_{\alpha x \leqq t \leqq \beta x} \left| \frac{L(t)}{L(x)} - 1 \right| = \sup_{\alpha \leqq \lambda \leqq \beta} \left| \frac{L(\lambda x)}{L(x)} - 1 \right| \to 0 \quad (x \to \infty) .$$

Hence

$$\lim_{x \to \infty} | \Psi_{[\alpha x,\beta x]} | = 0 . \tag{4.6.6}$$

From (4.6.3), (4.6.4), (4.6.5) and (4.6.6) it follows that

$$\limsup_{x \to \infty} \left| \Psi \left(\frac{L(t)}{L(x)} - 1, x \right) \right| \leqq (\alpha^\eta + \beta^{-\eta}) M ,$$

and (4.6.1) is proved by choosing α arbitrarily small and β arbitrarily large.

(Necessity). The necessity of (3.12) is obvious. As for (3.8) and (3.9), in view of the proof of Theorem 5, it will be sufficient to show that our hypothesis (3.11) implies (4.5.7).

Let $f \in \mathscr{M}_0$ be such that $\lim_{x\to\infty} f(x) = c$. If L is any SV function in $\mathscr{M}$, let

$$l(x) = (2\|f\| + f(x))\, L(x) .$$

The function l is clearly a SV function in $\mathscr{M}_0$ and we have

$$\Psi(f L, x) = \Psi(l, x) - 2\|f\|\, \Psi(L, x) .$$

If we define the operator Ψ_L by

$$\Psi_L(f, x) = \frac{1}{L(x)}\, \Psi(Lf, x) ,$$

then Ψ_L is a regular operator and

$$\Psi_L(f, x) = \frac{1}{L(x)}\, \Psi(f L, x)$$

$$= (2\|f\| + f(x))\, \frac{\Psi(l, x)}{l(x)} - 2\|f\|\, \frac{\Psi(L, x)}{L(x)} .$$

By (3.11) we have $\Psi(l, x)/l(x) \to 1$ and $\Psi(L, x)/L(x) \to 1$ $(x\to\infty)$ and so

$$\Psi_L(f, x) \to 2\|f\| + c - 2\|f\| = c \quad (x\to\infty) .$$

Hence, by Theorem 3, the operator Ψ_L preserves convergence and consequently

$$W_{\Psi_L}(1, x) = O(1) \quad (x\to\infty) .$$

But

$$W_{\Psi_L}(1, x) = \frac{1}{L(x)}\, W_\Psi(L, x)$$

and the necessity of (4.5.7) is proved.