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AN INTEGRAL INEQUALITY IN ANALYTIC FUNCTION THEORY

by Hiroshi HARUKI

The following theorem was proved in [1]:

THEOREM A. *Suppose that $f = f(z)$ is an entire function of a complex variable z . Then the only solutions of the functional inequality*

$$(1) \quad |f((x+y)/2)| \leq (|f(x)| + |f(y)|)/2,$$

where x, y are complex variables, are $f(z) = (Az + B)^n$ and $f(z) = \exp(Az + B)$ where A, B are arbitrary complex constants and n is an arbitrary positive integer.

Now, we shall prove that (1) implies the following integral inequality:

$$(2) \quad |1/(y-x) \int_C f(z) dz| \leq (|f(x)| + |f(y)|)/2,$$

where $f = f(z)$ is an entire function of z , x, y are complex variables ($x \neq y$) and C is an arbitrary contour joining two points x and y .

To this end we shall apply the following lemma, the easy proof of which is omitted:

LEMMA 1. If $k = k(t)$ is a real-valued continuous function of a real variable t and if k is convex on $[a, b]$, then

$$1/(b-a) \int_a^b k(t) dt \leq (k(a) + k(b))/2.$$

Now, we put $g(t) = |f(x+(y-x)t)|$, where $t \in [0, 1]$ and x, y are arbitrary distinct complex constants. $g(t)$ is a real-valued continuous function on $[0, 1]$. By (1) $g(t)$ is convex on $[0, 1]$. Hence, by Lemma 1 we have

$$\int_0^1 g(t) dt \leq (g(0) + g(1))/2.$$

Therefore we have

$$\int_0^1 |f(x + (y-x)t)| dt \leq (|f(x)| + |f(y)|)/2$$

and

$$(3) \quad |1/(y-x) \int_L f(z) dz| \leq (|f(x)| + |f(y)|)/2,$$

where L is the line segment joining the two points x and y . By Cauchy's Integral Theorem (3) implies (2).

The purpose of this note is to solve (2), i.e., to prove the following

THEOREM. *Suppose that $f = f(z)$ is an entire function of z . Then the only solutions of (2) are $f(z) = (Az+B)^n$ and $f(z) = \exp(Az+B)$ where A, B are arbitrary complex constants and n is an arbitrary positive integer.*

To this end we shall apply the following two lemmas:

LEMMA 2. (See [1].) Suppose that $f = f(z), g = g(z)$ are entire functions of z . If $|f(z)| \leq |g(z)|$ holds in $|z| < +\infty$, then $f(z) = Cg(z)$ where C is a complex constant with $|C| \leq 1$.

Proof. The proof is clear from Riemann's Theorem concerning a removable singularity and Liouville's Theorem.

LEMMA 3. Suppose that $H = H(z)$ is an entire function of z . If $A(t) = |H(t \exp(i\varphi))|^2$ where t, φ are real and φ is arbitrarily fixed, then we have

$$(i) \quad A''(0) = 2\operatorname{Re}(\exp(2i\varphi) H''(0) \overline{H(0)}) + 2 |H'(0)|^2,$$

$$(ii) \quad A^{(4)}(0) = 2\operatorname{Re}(\exp(4i\varphi) H^{(4)}(0) \overline{H(0)} \\ + 4 \exp(2i\varphi) H^{(3)}(0) \overline{H'(0)}) + 6 |H''(0)|^2.$$

Proof. Since the proof is easy, we omit it.

We may now prove our theorem.

Let $F = F(z)$ be an entire function such that $F'(z) = f(z)$. By (2) we have for all complex x, y

$$2 |F(y) - F(x)| \leq |y - x| (|f(x)| + |f(y)|).$$

By a corollary of Schwarz's Inequality ($(a+b)^2 \leq 2(a^2+b^2)$, a, b real) we have

$$2 |F(x) - F(y)|^2 \leq |x - y|^2 (|f(x)|^2 + |f(y)|^2).$$

Replacing x, y by $x + y, x - y$, respectively, we get

$$|F(x+y) - F(x-y)|^2 \leq 2|y|^2 (|f(x+y)|^2 + |f(x-y)|^2).$$

Putting $y = t \exp(i\varphi)$ where t, φ are real we have

$$\begin{aligned} & |F(x+t \exp(i\varphi)) - F(x-t \exp(i\varphi))|^2 \\ & \leq 2t^2 (|f(x+t \exp(i\varphi))|^2 + |f(x-t \exp(i\varphi))|^2). \end{aligned}$$

Keeping x, φ arbitrarily fixed and putting

$$\begin{aligned} p(t) = & 2t^2 (|f(x+t \exp(i\varphi))|^2 + |f(x-t \exp(i\varphi))|^2) \\ & - |F(x+t \exp(i\varphi)) - F(x-t \exp(i\varphi))|^2, \end{aligned}$$

$p(t)$ is a real-valued function of t and is of course four times differentiable on $|t| < +\infty$. Further $p(t)$ is an even function of t . Hence we have

$$(4) \quad p'(0) = 0, \quad p^{(3)}(0) = 0.$$

By Lemma 3 we have

$$\begin{aligned} (5) \quad p''(0) &= 8(|f(x)|^2 - |F'(x)|^2) = 0, \\ p^{(4)}(0) &= 96 \operatorname{Re}(\exp(2i\varphi) f''(x) \overline{f(x)}) + 96 |f'(x)|^2 \\ &\quad - 32 \operatorname{Re}(\exp(2i\varphi) F^{(3)}(x) \overline{F'(x)}) \\ &= 64 \operatorname{Re}(\exp(2i\varphi) f''(x) \overline{f(x)}) + 96 |f'(x)|^2. \end{aligned}$$

$p(t)$ has a minimum at $t = 0$ ($p(t) \geq 0$ on $|t| < +\infty, p(0) = 0$). Hence, by (4), (5) we have $p^{(4)}(0) \geq 0$, or

$$2 \operatorname{Re}(\exp(2i\varphi) f''(x) \overline{f(x)}) + 3 |f'(x)|^2 \geq 0.$$

x, φ were arbitrarily fixed. An appropriate choice of φ_0 gives

$$\operatorname{Re}(\exp(2i\varphi_0) f''(x) \overline{f(x)}) = -|f''(x) \overline{f(x)}| = -|f''(x) f(x)|.$$

Hence we have in $|x| < +\infty$

$$2|f''(x) f(x)| \leq 3|f'(x)|^2,$$

and by Lemma 2

$$(6) \quad f''(x) f(x) = K f'(x)^2,$$

where K is a complex constant with $|K| \leq 3/2$.

Solving (6) and taking into account the fact that f is an entire function, we have

$$(7) \quad f(z) = (Az + B)^n \quad \text{or} \quad f(z) = \exp(Az + B),$$

where A, B are complex constants and n is an arbitrary positive integer. By Theorem A, (7) satisfies (1). Since (1) implies (2) as already proved, (7) satisfies (2). Q.E.D.

REFERENCE

- [1] HIROSHI HARUKI. On the functional inequality $|f((x+y)/2)| \leq (|f(x)| + |f(y)|)/2$, *Journal of the Mathematical Society of Japan*, 16 (1964), pp. 39-41.

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