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A proof of Theorem 1 may be found on page 28 of Serre [4].

Recall that:

- (1)  $\deg K = 2g - 2$  where  $K$  denotes the canonical divisor on  $X$ ,
- (2) the Riemann-Roch theorem, i.e.  $l(D) = \deg D + 1 - g + l(K - D)$  where  $l(D) = \dim_k L(D)$ , and
- (3) if  $X$  is a non-singular plane curve of degree  $n$ , then  $g = (n-1)(n-2)/2$ .

*Def.*  $X$  is an *elliptic curve* if  $g = 1$ .

Notice that if  $D$  is a divisor of degree  $n$  on a curve  $X$ , then  $n < 0 \Rightarrow L(D) = 0 \Rightarrow l(D) = 0$ . In particular, on an elliptic curve  $X$ ,  $n > 0 \Rightarrow \deg(K - D) = -n < 0 \Rightarrow l(K - D) = 0 \Rightarrow l(D) = n$  from (1) and (2) above.

*Theorem 2* A non-singular complete curve  $C$  in  $\mathbf{P}^2$  of degree 3 is an elliptic curve.

*Proof:*

$$(3) \Rightarrow g = (3-1)(3-2)/2 = 1.$$

*Theorem 3* Every elliptic curve  $X$  is isomorphic to a non-singular complete irreducible curve  $C$  in  $\mathbf{P}^2$  of degree 3.

*Proof:*

Let  $D$  be a divisor of degree 3 on  $X$ .

Theorem 1 implies that  $D$  is very ample, i.e. that we have an isomorphism from  $X$  to a non-singular complete irreducible curve  $C$  in  $\mathbf{P}(L(D))$ . Riemann-Roch  $\Rightarrow l(D) = 3 \Rightarrow \mathbf{P}(L(D)) = \mathbf{P}^2$ . Let  $n = g(C)$ .  $X$  an elliptic curve  $\Rightarrow 1 = g(X) = g(C) = (n-1)(n-2)/2 \Rightarrow n = 3$ .

Thus we have established the desired connection between (I) and (II).

## § 2. ALGEBRAIC AND GEOMETRIC GROUP LAWS ON AN ELLIPTIC CURVE

Let  $X$  be an elliptic curve over  $k$ , and let  $X(k)$  denote the set of  $k$ -points of  $X$ . We begin by defining a group law on  $X(k)$  in a rather algebraic fashion. Let  $\text{Div}^0(X)$  be the group of divisors of degree 0 on  $X$ . Let  $\sim$  denote linear equivalence, and let  $\text{Div}^0(X)/\sim$  be the quotient group. If  $D \in \text{Div}^0(X)$ , let  $\text{Cl}(D)$  be its image in  $\text{Div}^0(X)/\sim$ .

Recall that a divisor  $D = \sum n_p P$  is called effective if  $n_p \geq 0$  for all  $P$ .

*Lemma 4* Let  $D_1$  and  $D_2$  be effective divisors of degree 1 on  $X$ . Then

$$(4) D_1 = D_2 \Leftrightarrow D_1 \sim D_2.$$

*Proof:*

( $\Rightarrow$ ) Obvious.

( $\Leftarrow$ )  $D_1$  effective  $\Rightarrow L(D_1)$  contains all the constant functions.  $\deg(D_1) = 1 \Rightarrow l(D_1) = 1 \Rightarrow L(D_1)$  consists solely of the constant functions. Suppose now that  $D_1 \sim D_2$ . Then there exists  $f \in k(X)$  such that  $D_1 + (f) = D_2$ .  $D_2$  effective  $\Rightarrow f \in L(D_1) \Rightarrow f$  constant  $\Rightarrow D_1 = D_2$ .

Fix a  $k$ -point  $e$  of  $X$ . Define a map  $\Phi$  from  $X(k)$  to  $\text{Div}^0(X)/\sim$  by  $P \rightarrow \text{Cl}(P-e)$ .

*Proposition 5* The map  $\Phi : X(k) \rightarrow \text{Div}^0(X)/\sim$  is a bijection.

*Proof:*

Claim  $\Phi$  is injective. Let  $P_1, P_2 \in X(k)$ .  $\Phi(P_1) = \Phi(P_2) \Leftrightarrow \text{Cl}(P_1 - e) = \text{Cl}(P_2 - e) \Leftrightarrow P_1 - e \sim P_2 - e \Leftrightarrow P_1 \sim P_2 \Leftrightarrow P_1 = P_2$ . So  $\Phi$  is injective. Claim  $\Phi$  is surjective. Let  $\bar{D} \in \text{Div}^0(X)/\sim$  with  $D \in \text{Div}^0(X)$  such that  $\text{Cl}(D) = \bar{D}$ .  $\deg(D+e) = 1 \Rightarrow l(D+e) = 1 \Rightarrow$  there exists  $f \in L(D+e)$ ,  $f \neq 0$ , such that  $(f) + D + e \geq 0$ , i.e.  $(f) + D + e = P$  for  $P \in X(k)$ .  $\Phi(P) = \text{Cl}(P-e) = \text{Cl}((f)+D) = \text{Cl}(D) = \bar{D}$ . Therefore  $\Phi$  is surjective, and hence bijective.

Thus  $X(k)$  receives an abelian group structure via  $\Phi$ , i.e. the sum of  $P_1$  and  $P_2$  is  $\Phi^{-1}(\Phi(P_1) + \Phi(P_2)) = \Phi^{-1}(\text{Cl}(P_1 - e) + \text{Cl}(P_2 - e)) = \Phi^{-1}(\text{Cl}(P_1 + P_2 - 2e)) =$  that point  $Q$  on  $X$  such that  $Q \sim P_1 + P_2 - e$ . We therefore have a map  $M : X(k) \times X(k) \rightarrow X(k)$  which we shall call the “algebraic” group law.

Now let us assume that  $C$  is a non-singular complete cubic in  $\mathbf{P}^2$ . We proceed to define a “geometric” group law on  $C(k)$ . If  $P_1, P_2 \in C(k)$ , there exists a unique line  $L$  such that the intersection cycle  $L \cdot C = P_1 + P_2 + P_3$  for some  $P_3 \in C(k)$ . If  $P_1 \neq P_2$ ,  $L$  is the unique line through  $P_1$  and  $P_2$ . If  $P_1 = P_2$ ,  $L$  is the unique tangent to  $C$  at  $P_1$ .  $P_3$  is thus uniquely determined by  $P_1$  and  $P_2$  and we have defined a mapping  $\varphi : C(k) \times C(k) \rightarrow C(k)$ . Let  $e$  be a fixed  $k$ -point of  $C$ . By repeating the preceding procedure with the points  $\varphi(P_1, P_2)$  and  $e$ , we will obtain a new point  $P_1 + P_2$ . Let  $m : C(k) \times C(k) \rightarrow C(k)$  be the resulting map, i.e.  $m$  is the composition of  $(e, \varphi)$  and  $\varphi$ ,  $m = \varphi^\circ(e, \varphi)$ .  $m$  is the “geometric” group law.

By using certain geometric properties of  $\mathbf{P}^2$ , it is possible to prove that  $m$  gives  $C(k)$  an abelian group structure (cf. Fulton [1], p. 125). We choose instead to prove the following proposition.

*Proposition 6* The “algebraic” group law on  $C$  coincides with the “geometric” group law on  $C$ , i.e.  $m = M$ .

*Proof:*

Let  $P_1, P_2 \in C(k)$ . Let  $P_3 = \varphi(P_1, P_2)$ . Then there exists a line  $L_1$  such that  $L_1 \cdot C = P_1 + P_2 + P_3$ . Let  $P_4 = \varphi(e, P_3) = \varphi(e, \varphi(P_1, P_2)) = m(P_1, P_2)$ . Then there exists a line  $L_2$  such that  $L_2 \cdot C = e + P_3 + P_4$ . Let  $f = L_1/L_2$  and regard  $f$  as an element of  $k(C)$ .  $(f) = P_1 + P_2 - e - P_4 \Rightarrow P_4 \sim P_1 + P_2 - e$ , i.e.  $P_4 = M(P_1, P_2)$ . Therefore  $m = M$ .

### § 3. ELLIPTIC CURVES AND ABELIAN VARIETIES

The purpose of this section is to prove the equivalence of notions (II) and (III). Up to this point, we have a group law on the set of  $k$ -points of an elliptic curve, and we would like to know that this is induced by an abelian variety structure. We shall also prove that 1-dimensional abelian varieties are elliptic curves.

*Definition* Let  $k$  be a field. An *abelian variety*  $X$  is a complete non-singular variety defined over  $k$  together with  $k$ -morphisms

$$\begin{aligned} m &: X \times X \rightarrow X \\ i &: X \rightarrow X \\ e &: \text{Spec}(k) \rightarrow X \end{aligned}$$

which satisfy the usual group axioms (cf. Mumford [2], p. 95).

To show that an elliptic curve can be given the structure of an abelian variety, it suffices to check that the map  $\varphi$  described in § 2 is a morphism. Recall that  $\varphi$  was defined on  $k$ -points as taking  $(P_1, P_2) \in C(k) \times C(k)$  to the unique third point  $P_3 \in C(k)$  such that  $P_1 + P_2 + P_3 = L \cdot C$  for some line  $L$ . It is quite easy to see that  $\varphi$  is a morphism on a certain affine open subset of  $C \times C$ . To be precise, we have the following lemma.

*Lemma 7*  $\varphi$  defines a morphism from

$$\mathcal{S} = \text{Spec}(k[X_1, Y_1, X_2, Y_2]/(f(X_1, Y_1), f(X_2, Y_2))(X_1 - X_2))$$

to  $\mathcal{T} = \text{Spec}(k[X_3, Y_3]/f(X_3, Y_3))$  (where  $f$  is an affine equation for  $C$ ).