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8. Case of a manifold with boundary

More generally we consider a closed manifold N of dimension p in a manifold M of dimension n. $L_{M,N}$ will denote the subalgebra of L_M of those vector fields on M which are tangent to N. An interesting particular case is when N is the boundary ∂M of M. For M compact, $L_{M,\partial M}$ can be considered as the Lie algebra of the group of diffeomorphisms of M.

First we consider the formal vector fields. *Let* $\mathfrak{a}_{n,p}$ *be the Lie subalgebra of formal vector fields on* R^n *which are tangent to* R^p identified to a linear subspace of R^n. Again $C^*(\mathfrak{a}_{n,p})$ denotes the DG-algebra of those multilinear alternate forms on $\mathfrak{a}_{n,p}$ depending only on finite order jets.

We describe a finite dimensional model for $C^*(\mathfrak{a}_{n,p})$. Let $E(h'_1, \ldots, h'_p, h''_1, \ldots, h''_{n-p})$ be the exterior algebra in generators h'_i and h''_j of degree $2i-1$. Let $R[c'_1, \ldots, c'_p, c_1'', \ldots, c''_{n-p}]^{\wedge}_{2p}$ be the quotient of the polynomial algebra in generators c'_i and c_i'' of degree $2i$ by the ideal of elements of degree $> 2p$.

Define

$$WU_{n,p} = E(h'_1, \ldots, h'_p, h''_1, \ldots, h''_{n-p}) \otimes R[c'_1, \ldots, c'_p, c''_1, \ldots, c''_{n-p}]^{\wedge}_{2p}$$

as the DG-algebra with differential defined by

$$dh_i' = c_i', \quad dh_i'' = c_i'', \quad dc_i' = 0, \quad dc_i'' = 0\,.$$

This is a model for the space $F_{n,p}$ obtained by restricting the universal principal $(U_p \times U_{n-p})$-bundle over the $2p$-skeleton of its basis represented by a product of Grassmanians with the usual even dimensional cell decomposition.

If $n \leqslant 2p$, $WU_{n,p}$ is also a model for a wedge of spheres. When $n > 2p$, it is a model for the product of the wedge of spheres corresponding to $WU_{2p,p}$ by $S^{2p+1} \times S^{2p+3} \ldots \times S^{2n-2p-1}$.

Theorem 1 (Koszul [11]). *There is a natural morphism*

$$WU_{n,p} \to C^*(\mathfrak{a}_{n,p})$$

inducing an isomorphism in cohomology.

As a consequence, $H^i(\mathfrak{a}_{n,p}) = 0$ for $0 < i \leqslant 2p$ and $i > p^2 + (n-p)^2 + 2p$. When $n \leqslant 2p$, the multiplication is trivial.

To have a model for the homomorphism induced by the inclusion of $\mathfrak{a}_{n,p}$ in $\mathfrak{a}_n$, we have the commutative diagramm

$$\begin{array}{ccc} C^*(\mathfrak{a}_n) & \longrightarrow & C^*(\mathfrak{a}_{n,p}) \\ \uparrow & & \uparrow \\ WU_n & \longleftarrow & WU_{n,p} \end{array}$$

where the second horizontal map sends h_i on $h_i' + h_i''$ and c_i on $c_i' + c_i''$ (by convention, h_i' or h_i'' is zero for $i > p$ or $i > n\text{-}p$, idem for c_i' and c_i''). Note that the natural map of theorem 1 should map the c_i' s and c_i'' not on the usual Chern classes defined by the connection but on the polynomials in Chern classes corresponding to $\sum x_k^i$, the Chern classes being the elementary symmetric functions in the formal variables x_k. These horizontal maps are also models for an inclusion of $F_{n,p}$ in F_n.

We consider again the bundle E over M associated to the tangent bundle of M and with fiber F_n. Its restriction above N contains a subbundle E' with fiber $F_{n,p}$.

THEOREM. *$C^*(L_{M,N})$ is a model for the space $\Gamma_{M,N}$ of continuous sections of the bundle E whose restriction to N have values in the subbundle E'.*

To make explicit computations, we construct a model for $\Gamma_{M,N}$, which will be finite dimensional in each degree when M and N have finite dimensional models. This is the purpose of the next paragraph.

9. CONSTRUCTION OF A MODEL FOR $C^*(L_{M,N})$

Consider the commutative diagramm of Lie algebras

$$\begin{array}{ccc} L_{M,N} & \longrightarrow & L_M \\ \downarrow & & \downarrow \\ L'_{M,N} & \longrightarrow & L'_M \end{array}$$

where L'_M and $L'_{M,N}$ are the quotients of L_M and $L_{M,N}$ by the subalgebra L_M^0 of vector fields on M whose infinite jet vanish at points of N.