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**Autor:** Glover, Henry H. / Mislin, Guido  
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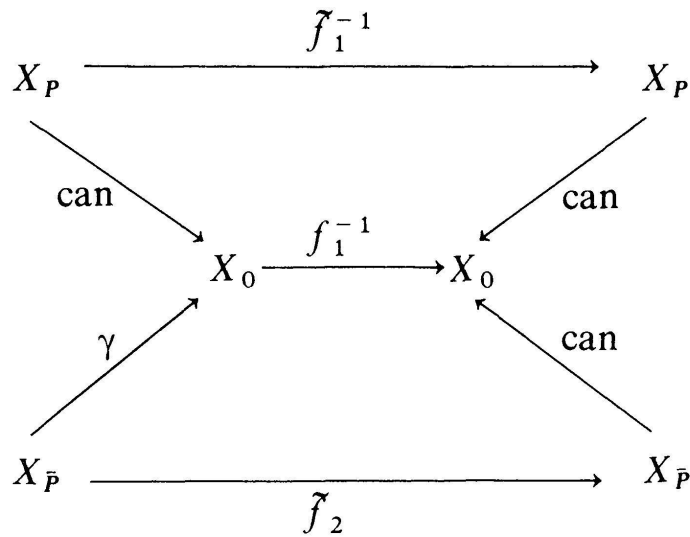
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which shows that  $Y \simeq X$ .

§2. THE CASE OF GENERALIZED FLAG MANIFOLDS

The following result is an easy consequence of [F].

LEMMA 2.1. Let  $M$  be a generalized flag manifold. Then the following holds.

- a) If  $g(\lambda) \in Gr(M_0)$  is a grading map with  $\lambda \in \mathbf{Z}_Q^*$  for some (not necessarily finite) set of primes  $Q$ , then  $g(\lambda)$  lifts to a homotopy equivalence  $\tilde{g}(\lambda): M_Q \rightarrow M_Q$ .
- b) Let  $P$  be an arbitrary set of primes with complement  $\bar{P}$ . Then every

$$f \in \langle Gr(M_0), N(H)/H \rangle$$

may be written in the form  $f = f_1 \circ f_2$  with

$$f_1 \in \text{im}(E(M_P) \rightarrow E(M_0))$$

and

$$f_2 \in \text{im}(E(M_{\bar{P}}) \rightarrow E(M_0)).$$

*Proof.* Let  $\lambda = k/l$  with  $k$  and  $l$  relatively prime integers. Then  $g(k)$  and  $g(l)$  lift to equivalences

$$\tilde{g}(k), \tilde{g}(l): M_Q \rightarrow M_Q.$$

since necessarily  $k, l \in \mathbf{Z}_Q^*$  (compare [F]). Thus  $\tilde{g}(k) \tilde{g}(l)^{-1}$  is a lift of  $g(\lambda)$ . For b) we note that  $f = g(\rho) \circ \sigma$  for some  $\rho \in \mathbf{Q}^*$  and

$$\sigma \in N(H)/H.$$

If we write  $\rho = \rho_1 \cdot \rho_2$  with  $\rho_1 \in \mathbf{Z}_P^*$  and  $\rho_2 \in \mathbf{Z}_P^*$ , then

$$f = g(\rho_1) \cdot (g(\rho_2) \sigma)$$

and we may choose

$$f_1 = g(\rho_1), f_2 = g(\rho_2) \sigma.$$

Since  $\sigma$  lifts even to  $E(M)$ , we infer by using a) that  $f_1$  and  $f_2$  lift as desired.

A final step towards proving the Theorem formulated in the introduction consists in the following.

LEMMA 2.2. Let  $M$  be a generalized flag manifold for which Conjecture C holds. Then for every finite set of primes  $P$ ,

$$P\text{-Seq}(E(M_0)) = \{[1, 1, \dots, 1]\}.$$

*Proof.* Let  $\{[\mu_1, \dots, \mu_n]\} \in P\text{-Seq}(E(M_0))$ , where  $P = \{p_1, \dots, p_n\}$  and

$$\mu_i \in \text{im}(E(M_{p_i}) \rightarrow E(M_0))$$

for all  $i$ . Then  $\mu_i = g(\lambda_i) \circ \sigma_i$  with  $\lambda_i \in \mathbf{Q}^*$  and

$$\sigma_i \in N(H)/H \subset E(M_0).$$

Define  $\lambda \in \mathbf{Q}^*$  by  $\lambda = \prod p_i^{m_i}$ , where  $m_i \in \mathbf{Z}$  is such that  $p_i^{m_i} \lambda_i \in \mathbf{Z}_{p_i}^*$ . Then  $g(\lambda) \mu_i = g(\lambda \lambda_i) \sigma_i$  with  $\lambda \lambda_i \in \mathbf{Z}_{p_i}^*$ . By Lemma 2.1 a) we know that  $g(\lambda \lambda_i)$  lifts to  $M_{p_i}$ , and since  $\sigma_i$  lifts even to  $M$  we conclude that

$$h(p_i) = g(\lambda \lambda_i) \sigma_i \in \text{im}(E(M_{p_i}) \rightarrow E(M_0))$$

for all  $i$ . The equation

$$g(\lambda) \mu_i = h(p_i), i \in \{1, \dots, n\}$$

show that  $\{[\mu_1, \dots, \mu_n]\} = \{[1, \dots, 1]\} \in P\text{-Seq}(E(M_0))$ .

The proof of the main Theorem:

Let  $M$  be a generalized flag manifold for which the Conjecture C holds. Since  $M$  is a formal space we can find for every  $N \in G(M)$  a rational equivalence

$f(N): N \rightarrow M$ . Let  $P(M)$  denote the set of primes which appear in any of the orders of

$$\ker(f(N)_*: H_*(N; \mathbf{Z}) \rightarrow H_*(M; \mathbf{Z}))$$

or  $\text{coker } f(N)_*$ ,  $N$  ranging over  $G(M)$ . The set  $P(M)$  is finite, since each  $\ker f(N)_*$  and  $\text{coker } f(N)_*$  is finite and since  $G(M)$  is a finite set by [W]. Consider now the map

$$\theta: G(M) \rightarrow P\text{-Seq } E(M_0)$$

with respect to this finite set of primes  $P(M) = P$ . Since  $P$  is finite,

$$P\text{-Seq } (E(M_0))$$

consists of only one element (Lemma 2.2). It remains to show that

$$\theta^{-1}(\theta(M)) = \{M\}.$$

For this we apply Lemma 1.3. Note that  $N \in G(M)$  implies  $N_{\bar{P}} \simeq M_{\bar{P}}$  since  $f(N): N \rightarrow M$  is a  $\bar{P}$ -equivalence. Moreover, the condition b) of 1.3 is satisfied in view of Lemma 2.1 b). Therefore we conclude that  $G(M) = \{[M]\}$  and the proof is completed.

*Note added in proof.* Since this paper went to press, we have been informed that Conjecture C has been proved for the case  $k = 2, n_1 = n_2$ , by M. Hoffman: "Cohomology endomorphisms of complex flag manifolds", Ph.D. dissertation, MIT 1981. As a consequence, it follows that all complex Grassmann manifolds are generically rigid.