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## §1. PRELIMINARIES

Let $G$ be a connected real semi-simple Lie group with finite center, $K$ a maximal compact subgroup of $G$, and let $\mathfrak{g} \supset \mathfrak{k}$ be the corresponding Lie algebras. For any sub-algebra $\mathfrak{a} \subset \mathfrak{g}$, we put

$$\mathfrak{a}_{\mathbf{C}} = \mathfrak{a} \otimes_{\mathbf{R}} \mathbf{C}.$$

If $B$ denotes the Killing form of $\mathfrak{g}$, $B$ is negative-definite on $\mathfrak{k}$, and we let $\mathfrak{p}$ denote the orthogonal complement under $B$ of $\mathfrak{k}$ in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a so-called Cartan decomposition of $\mathfrak{g}$, and $B$ is positive-definite on $\mathfrak{p}$.

Let $M = G/K$, the corresponding symmetric space. As $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $B$ defines an $(\mathrm{Ad}\ K)$-invariant inner product on $\mathfrak{p}$; and since we may identify $\mathfrak{p}$ naturally as the tangent space to $M$ at the identity coset $x_0 = K$, $B$ determines a unique Riemannian metric on $M$ which is invariant under the canonical left $G$-action.

Assume initially that $M$ is an *irreducible* symmetric space. Then, if one wishes, $G$ can be taken to be a non-compact almost simple group (i.e., $\mathfrak{g}$ is a simple Lie algebra). In that case, the space $M$ admits a homogeneous complex structure, and becomes an *Hermitian* symmetric space, precisely when $\mathfrak{k}$ has a non-trivial center $\mathfrak{z}$. In this case, $\dim \mathfrak{z} = 1$, and $Z = \exp \mathfrak{z}$ is the identity component of the center of $K$. Let $G^{\mathrm{ad}}$ denote the adjoint group of $G$ (i.e., the automorphism group of $M$) and let $K^{\mathrm{ad}}$, $Z^{\mathrm{ad}}$ be the corresponding subgroups of $G^{\mathrm{ad}}$. A choice of $z_0 \in Z^{\mathrm{ad}}$ of order 4 (for which $\mathrm{Ad}(z_0^2)$ is a Cartan involution of $\mathfrak{g}$) determines an almost-complex structure on $\mathfrak{p}$:

$$\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}^{+} \oplus \mathfrak{p}^{-}$$

with

(1.1) $$\begin{cases} \mathfrak{p}^{+} = \{X \in \mathfrak{p}_{\mathbf{C}} : \operatorname{Ad}(z_0)\, X = iX\} \\ \mathfrak{p}^{-} = \{X \in \mathfrak{p}_{\mathbf{C}} : \operatorname{Ad}(z_0)\, X = -iX\} \end{cases}$$

This determines, via left-translation under $G$, a Kählerian complex structure on $M$, such that the action of $G$ is by holomorphic isometries.

For purposes of numeration, we define $\mu = \mu(G)$ to be the degree of the covering map $Z \to Z^{ad}$. It has the following properties:

(1.2) i) if $G$ is of adjoint type, $\mu = 1$ (cf. [16, (1.17B)]),

ii) if $G' \to G$ is a finite covering, then $\mu(G)$ divides $\mu(G')$,

iii) if $G = SU(n, 1)$, then $\mu = n + 1$.

Let $\rho$ be an irreducible representation of $G$ on the finite-dimensional complex vector space $V$. We will say that $(\rho, V)$ is a *real* representation if there is a $G$-invariant $\mathbf{R}$-subspace $V_{\mathbf{R}}$ of $V$ with

$$V = V_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C},$$

such that $G$ acts on $V$ by extension of scalars. Under the subgroup $Z$, the representation necessarily splits into one-dimensional $Z$-invariant summands, on each of which $Z$ acts by a character. We pick an isomorphism

(1.3) $$\phi : Z \simeq S^1 = \{w \in \mathbf{C} : |w| = 1\}.$$

The character group of $Z$ is free cyclic, with elements

$$\chi_n : Z \to S^1$$

given by

$$\chi_n(z) = [\phi(z)]^n.$$

Let

(1.4) $$V\langle n\rangle = \{v \in V : \rho(z)\, v = \chi_n(z)\, v \quad \text{if} \quad z \in Z\},$$

so that

(1.5) $$V = \oplus_{n \in \mathbf{Z}} V\langle n\rangle.$$

Each $V\langle n\rangle$ is invariant under $K$. If $(\tau, W)$ is a representation of $K$, then we define $W\langle n\rangle$ as in (1.4); if $W$ is irreducible, then $W = W\langle n\rangle$ for some $n$, i.e., $Z$ acts by a single character. We will assume to have chosen the isomorphism (1.3) so that $Z$ acts on $\mathfrak{p}^+$ by a "positive" character.

(1.6) *Example.* Assuming that $G$ is almost simple, we take $V_{\mathbf{R}} = \mathfrak{g}$, and $\rho = \mathrm{Ad}$, the adjoint representation of $G$. Then $\mathfrak{p}^+ = V<\mu>$, $\mathfrak{k}_{\mathbf{C}} = V<0>$, and $\mathfrak{p}^- = V<-\mu>$.

For an irreducible (finite-dimensional) representation of $G$, we also use $\rho$ to denote the induced action of $\mathfrak{g}$ on $V$. Because of the above description (1.6) of Ad, it is easy to see that the following hold:

(1.7) i) $\rho(\mathfrak{p}^+)\, V<n> \subset V<n+\mu>$, $\rho(\mathfrak{k})\, V<n> \subset V<n>$, $\rho(\mathfrak{p}^-)\, V<n> \subset V<n-\mu>$.

ii) $\{n : V<n> \neq 0\} = \{\lambda, \lambda - \mu, \lambda - 2\mu, \ldots, \lambda - m\mu\}$ for some integers $\lambda \geqslant 0$, $m \geqslant \mu^{-1}\lambda$.

iii) If $V$ is real, then for all $n$,

$$V<-n> = \overline{V<n>} \qquad \text{(complex conjugate)}$$

and thus $m\mu = 2\lambda$.

((1.7 i) includes, in particular, the standard fact that $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are Abelian Lie subalgebras of $\mathfrak{g}_{\mathbf{C}}$.)

For the general case, write

$$G = (\Pi_{j=1}^{l} G_j) \times H, \tag{1.8}$$

where each $G_j$ is almost simple and of non-compact Hermitian type, and $H$ is compact.[1]) Let

$$K = (\Pi_{j=1}^{l} K_j) \times H;$$

$Z = \Pi_{j=1}^{l} Z_j$; and $Z^{\mathrm{ad}} = \Pi_{j=1}^{l} Z_j^{\mathrm{ad}}$, a product of circles. Let $\Delta^{\mathrm{ad}}$ be the diagonal of $Z^{\mathrm{ad}}$, and $\Delta$ the inverse image of $\Delta^{\mathrm{ad}}$ in $Z$. One may proceed as before, if we replace $Z$ by $\Delta$. Alternatively, every irreducible representation $(\rho, V)$ of $G$ decomposes as a tensor product

$$(\otimes_{j=1}^{l} (\rho_j, V_j)) \otimes (\sigma, W)$$

in accordance with the product structure (1.8). It is then easy to see that under the action of $\Delta$, the decomposition (1.5) of $V$ is the tensor product of the

[1]) We allow compact factors because of (2.7).

corresponding decompositions of each $V_j$ into character spaces under $Z_j$, tensored with the "trivial" factor $W$.

On $V$ there exists a positive-definite Hermitian form (the *admissible inner product*) $T(v, w)$ (see [7, p. 375]), determined uniquely up to a constant multiple, with the property that

$$\text{(1.9)} \quad \text{i) } T(\rho(k)v, \rho(k)w) = T(v, w) \quad \text{if } k \in K$$
$$\text{ii) } T(\rho(X)v, w) = T(v, \rho(X)w) \quad \text{if } X \in \mathfrak{p}.$$

This follows from the fact that $\mathfrak{k} \oplus i\mathfrak{p}$ is a compact Lie algebra. If $V$ is real, then the admissible inner product can be seen to be the Hermitian extension of a real inner product on $V_{\mathbf{R}}$.

Let $I$ denote the intersection of the kernels of all finite-dimensional representations of $G$. Then $I$ is a central subgroup, and $G/I$ admits the structure of a real (affine) algebraic group. Since we are interested in $G$ only for its finite-dimensional representations and the symmetric space $M$, we may replace $G$ by $G/I$ and assume that $G$ is an algebraic group. To get all of the representations of $\mathfrak{g}$, it is convenient in the abstract to replace $G$ by its algebraic universal covering group (i.e., one makes the preceding construction for the topological universal cover of $G$); thus, *we may and do assume that* $G$ *is algebraically simply connected.* We remark that by (1.2), the number $\mu(G)$ can be arbitrarily large, even under this restriction.

Let, then, $G_{\mathbf{C}}$ denote the set of complex points of $G$. It is a simply-connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbf{C}}$. Let $K_{\mathbf{C}}$ denote the connected subgroup of $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{k}_{\mathbf{C}}$. By general theory (see [17, XVII.5]), $K_{\mathbf{C}}$ is the universal complexification of $K$, and so, by definition, every representation of $K$ extends to a holomorphic representation of $K_{\mathbf{C}}$.

Assume that $M$ is Hermitian, and let $P^+$ (resp. $P^-$) denote the subgroup of $G_{\mathbf{C}}$ corresponding to the subalgebra $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) of $\mathfrak{g}_{\mathbf{C}}$. Then $P^+K_{\mathbf{C}}P^-$ is an open subset of $G_{\mathbf{C}}$ which contains $G$ (see [4, p. 317]). Moreover, $G \cap K_{\mathbf{C}} P^- = K$ (see [4, p. 318]), so the mapping of $G \to G_{\mathbf{C}}$ induces a holomorphic embedding

$$\text{(1.10)} \qquad M \to \check{M} = G_{\mathbf{C}}/Q;$$

as $Q = K_{\mathbf{C}} P^-$ is a parabolic subgroup of $G_{\mathbf{C}}$, $\check{M}$ is compact and is known as the *compact dual* of $M$.