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III. A VARIANT

Let us agree to call a scheme S *accessible* if there exists an absolutely finitely generated field K for which the set $S(K)$ of K -valued points of S is non-empty. Thus for example, if K is an absolutely finitely generated field, then for *any* subring $R \subset K$, $\text{Spec}(R)$ is accessible (by the K -valued point $R \hookrightarrow K$); also any subring R' of the power-series ring $K[[X_1, \dots, \dots]]$ over K in any number of variables has $\text{Spec}(R')$ accessible

$$\text{(by } R' \hookrightarrow K[[X_1, \dots]] \xrightarrow{X \rightarrow 0} K \text{)}.$$

On the other hand, the spectrum of a field F is accessible if and only if F is absolutely finitely generated.

THEOREM 2. *Let S be a connected, locally noetherian scheme which is accessible. Let X/S be a proper and smooth S -scheme with geometrically connected fibres. Then the group $\text{Ker}(X/S)$ is finite.*

Proof. We begin by reducing to the case when S is a finitely generated field. In view of the accessibility of S , this reduction results from the following simple lemma applied with $T = \text{Spec}(K)$.

LEMMA 4. *Let X/S be proper and smooth with geometrically connected fibres over a connected locally noetherian scheme S . Given a connected locally noetherian S -scheme T , denote by X_T/T the inverse image of X/S on T , i.e. form the cartesian diagram*

$$\begin{array}{ccc} & X_T = X \times_S T & \\ & \swarrow & \downarrow \\ X & & T \\ \downarrow & \swarrow & \\ S & & \end{array}$$

The natural map (cf. 1.5)

$$\text{Ker}(X_T/T) \rightarrow \text{Ker}(X/S)$$

is surjective.

Proof. Let t be a geometric point of T , s the image geometric point of S , and x a geometric point on the fibre X_s . The homotopy exact sequences (SGA I, Exp X, 1.4) for X/S and X_T/T sit in a commutative diagram

$$\begin{array}{ccccccc} \pi_1(X_s, x) & \longrightarrow & \pi_1(X_T, x) & \longrightarrow & \pi_1(T, t) & \longrightarrow & 0 \\ \parallel & & \downarrow & & \downarrow & & \\ \pi_1(X_s, x) & \longrightarrow & \pi_1(X, x) & \longrightarrow & \pi_1(S, s) & \longrightarrow & 0 \end{array}$$

Passing to the abelianizations yields the commutative diagram with exact rows

$$\begin{array}{ccccccc} \pi_1(X_s)^{ab} & \longrightarrow & \pi_1(X_T)^{ab} & \longrightarrow & \pi_1(T)^{ab} & \longrightarrow & 0 \\ \parallel & & \downarrow & & \downarrow & & \\ \pi_1(X_s)^{ab} & \longrightarrow & \pi_1(X)^{ab} & \longrightarrow & \pi_1(S)^{ab} & \longrightarrow & 0 \end{array}$$

whence we find

$$\begin{array}{l} \pi_1(X_s)^{ab} \begin{cases} \longrightarrow \text{Ker}(X_T/T) = \text{image of } \pi_1(X_s)^{ab} \text{ in } \pi_1(X_T)^{ab} . \\ \longrightarrow \text{Ker}(X/S) = \text{image of } \pi_1(X_s)^{ab} \text{ in } \pi_1(X)^{ab} . \end{cases} \end{array} \quad \text{QED}$$

Thus we are reduced to proving the finiteness of $\text{Ker}(X/K)$ when K is an absolutely finitely generated field, and X/K is proper, smooth, and geometrically connected. We have already proven this finiteness theorem when X/K is an abelian variety (cf. Remark (1) above). We will reduce to this case by making use of the theory of the Picard and Albanese varieties.

At the expense of replacing K by a finite extension, we may assume that X has a K -rational point x_0 . The Picard scheme $\text{Pic}_{X/K}$ is then a commutative group-scheme locally of finite type over K , which represents the functor on $\{\text{Schemes}/K\}$

$$W \rightarrow \left\{ \begin{array}{l} \text{the group of } W\text{-isomorphism classes of pairs } (\mathcal{L}, \varepsilon) \text{ consisting} \\ \text{of an invertible sheaf } \mathcal{L} \text{ on } X \times_K W \text{ together with a} \\ \text{trivialization } \varepsilon \text{ of the restriction } \mathcal{L} \text{ to } \{x_0\} \times_K W \end{array} \right.$$

The subgroup-scheme $Pic_{X/K}^\tau$ of $Pic_{X/K}$ classifies those $(\mathcal{L}, \varepsilon)$ whose underlying \mathcal{L} becomes τ -equivariant to zero when restricted to every geometric fibre of $X \times W/W$ (i.e. for each geometric point w of W , some multiple of $\mathcal{L} \mid X \times w$ is algebraically equivalent to zero). The identity component $Pic_{X/K}^0$ of $Pic_{X/K}$ classifies those $(\mathcal{L}, \varepsilon)$ whose \mathcal{L} becomes algebraically equivalent to zero on each geometric fibre $X \times W/W$. The Picard variety $Pic_{X/K}^{0, \text{red}}$ is an abelian variety over K , and it sits in an *f.p.p.f.* short exact sequence of commutative group schemes

$$(3.1) \quad 0 \rightarrow Pic_{X/K}^{0, \text{red}} \rightarrow Pic_{X/K}^\tau \rightarrow C \rightarrow 0$$

in which the cokernel C is a finite flat group-scheme over K . This cokernel C should be thought of as the “scheme theoretic” torsion in the Neron-Severi group.

We denote by $Alb_{X/K}$ the Albanese variety of X/K , defined to be the dual abelian variety to the Picard variety $Pic_{X/K}^{0, \text{red}}$. We now recall the expression of $\pi_1(X \otimes \bar{K})^{ab}$ in terms of the Tate module of the Albanese, and a finite “error term” involving the Cartier dual C^\vee of C .

LEMMA 5. *Let K be a field, and X/K a proper, smooth and geometrically connected K -scheme which admits a \bar{K} -rational point. Then there is a canonical short exact sequence of $\text{Gal}(\bar{K}/K)$ -modules*

$$(3.2) \quad 0 \rightarrow C^\vee(\bar{K}) \rightarrow \pi_1(X \otimes \bar{K})^{ab} \rightarrow T(Alb_{X/K}(\bar{K})) \rightarrow 0.$$

Proof. By Kummer and Artin-Schreier theory, we have for each integer $N \geq 1$ a canonical isomorphism

$$\begin{aligned} & \text{Hom}(\pi_1(X \otimes \bar{K})^{ab}, \mathbf{Z}/N\mathbf{Z}) \\ &= H_{\text{et}}^1(X \otimes \bar{K}, \mathbf{Z}/N\mathbf{Z}) \simeq \text{Hom}(\mu_N, (Pic_{X/K}^\tau) \otimes \bar{K}). \end{aligned}$$

in which the last Hom is in the sense of \bar{K} -group-schemes. Applying the functor $X \mapsto \text{Hom}(\mu_N, X)$ to the short exact sequence

$$0 \rightarrow Pic^{0, \text{red}} \rightarrow Pic^\tau \rightarrow C \rightarrow 0$$

gives a short exact sequence

$$(3.3) \quad \begin{aligned} 0 & \rightarrow \text{Hom}(\mu_N, (Pic^{0, \text{red}}) \otimes \bar{K}) \\ & \rightarrow \text{Hom}(\mu_N, (Pic^\tau) \otimes \bar{K}) \rightarrow \text{Hom}(\mu_N, C \otimes \bar{K}) \rightarrow 0 \end{aligned}$$

(the final zero because over an algebraically closed field, the group $\text{Ext}^1(\mu_N, A)$ vanishes for any abelian variety A , cf. the remark at the end of this section). We now “decode” its two end terms, using Cartier-Nishi duality for the first, and Cartier duality for the last.

The first is

$$\begin{aligned} \text{Hom}(\mu_N, (\text{Pic}^{0, \text{red}}) \otimes \bar{K}) &= \text{Hom}(\mu_N, (\text{Pic}^{0, \text{red}})_N \otimes \bar{K}) \\ &\quad \Downarrow \text{Cartier-Nishi duality} \\ &\text{Hom}(\text{Alb}_{X/N})_N \otimes \bar{K}, \mathbf{Z}/N\mathbf{Z} \\ &\quad \Downarrow \text{evaluation on } \bar{K}\text{-points} \\ &\text{Hom}((\text{Alb}_{X/K}(\bar{K}))_N, \mathbf{Z}/N\mathbf{Z}) \\ &\quad \Downarrow \\ &\text{Hom}(T(\text{Alb}_{X/K}(\bar{K})), \mathbf{Z}/N\mathbf{Z}). \end{aligned}$$

The last is

$$\begin{aligned} \text{Hom}(\mu_N, C \otimes \bar{K}) &\xrightarrow{\text{Cartier duality}} \text{Hom}(C^\vee \otimes \bar{K}, \mathbf{Z}/N\mathbf{Z}) \\ &\quad \Downarrow \text{evaluation} \\ &\text{Hom}(C^\vee(\bar{K}), \mathbf{Z}/N\mathbf{Z}) \end{aligned}$$

“Substituting” into the exact sequence (3.2), we find a canonical short exact sequence

$$(3.4) \quad \begin{aligned} 0 &\rightarrow \text{Hom}(T(\text{Alb}_{X/K}(\bar{K})), \mathbf{Z}/N\mathbf{Z}) \\ &\rightarrow \text{Hom}(\pi_1(X \otimes \bar{K})^{ab}, \mathbf{Z}/N\mathbf{Z}) \rightarrow \text{Hom}(C^\vee(\bar{K}), \mathbf{Z}/N\mathbf{Z}) \rightarrow 0 \end{aligned}$$

Passing to the *direct* limit as N grows multiplicatively, we obtain a canonical short exact sequence

$$(3.5) \quad \begin{aligned} 0 &\rightarrow \text{Hom}(T(\text{Alb}_X(\bar{K})), \mathbf{Q}/\mathbf{Z}) \\ &\rightarrow \text{Hom}(\pi_1(X \otimes \bar{K})^{ab}, \mathbf{Q}/\mathbf{Z}) \rightarrow \text{Hom}(C^\vee(\bar{K}), \mathbf{Q}/\mathbf{Z}) \rightarrow 0. \end{aligned}$$

Taking its Pontryagin dual, we find the required exact sequence (3.2). QED

To complete the reduction of Theorem 2 to the case of abelian varieties, we simply notice that the exact sequence of lemma 5 yields, upon passage to coinvariants, an exact sequence

$$(3.6) \quad (C^\vee(\overline{K}))_{\text{Gal}(\overline{K}/K)} \rightarrow \text{Ker}(X/K) \rightarrow \text{Ker}(\text{Alb}_{X/K}/K) \rightarrow 0$$

whose first term, being a quotient of the finite group $C^\vee(\overline{K})$, is finite. QED

Remark. In the course of the proof of Lemma 5, we appealed to the “well-known” vanishing of $\text{Ext}^1(\mu_N, A)$ over an algebraically closed field, for an abelian variety A and any integer $N > 1$. Here is a simple proof. It is enough to prove this vanishing when N is either prime to the characteristic p of K , or, in case $p > 0$, when $N = p$.

Suppose first N prime to p . Because the ground-field is algebraically closed, we have $\mu_N \simeq \mathbf{Z}/N\mathbf{Z}$, so it is equivalent to prove the vanishing of $\text{Ext}^1(\mathbf{Z}/N\mathbf{Z}, A)$. We will prove that *this* group vanishes for every integer $N > 1$. Consider such an extension:

$$0 \rightarrow A \rightarrow E \rightarrow \mathbf{Z}/N\mathbf{Z} \rightarrow 0$$

Pass to \overline{K} -valued points

$$0 \rightarrow A(\overline{K}) \rightarrow E(\overline{K}) \rightarrow \mathbf{Z}/N\mathbf{Z} \rightarrow 0$$

and consider the endomorphism “multiplication by N ”. Because the group $A(\overline{K})$ is N -divisible, the snake lemma gives an exact sequence

$$0 \rightarrow A(\overline{K})_N \rightarrow E(\overline{K})_N \rightarrow \mathbf{Z}/N\mathbf{Z} \rightarrow 0$$

But a point in $E(\overline{K})_N$ which maps onto “1” $\in \mathbf{Z}/N\mathbf{Z}$ is precisely a splitting of our extension.

Next consider the case $N = p = \text{char}(K)$. We give a proof due to Barry Mazur. Using the *f.p.p.f.* exact sequence

$$0 \rightarrow A_p \rightarrow A \rightarrow A \rightarrow 0.$$

to compute $\text{Ext}(\mu_p, -)$, we obtain a short exact sequence

$$0 \rightarrow \text{Hom}(\mu_p, A) \rightarrow \text{Ext}^1(\mu_p, A_p) \rightarrow \text{Ext}^1(\mu_p, A) \rightarrow 0$$

To prove that $\text{Ext}^1(\mu_p, A) = 0$, we will show that the groups $\text{Hom}(\mu_p, A)$ and $\text{Ext}^1(\mu_p, A_p)$ are both finite, of the same order. Trivially, we have $\text{Hom}(\mu_p, A) = \text{Hom}(\mu_p, A_p)$. Because we are over an algebraically closed field, and A_p is killed by p , its toroidal biconnected-etale decomposition looks like

$$A_p \simeq (\mu_p)^a \times (\text{biconnected}) \times (\mathbf{Z}/p\mathbf{Z})^b; \quad [\text{in fact } a = b].$$

Only the μ_p 's in A_p can “interact” with μ_p . Thus we are reduced to showing that $\text{Hom}(\mu_p, (\mu_p)^a)$ and $\text{Ext}^1(\mu_p, (\mu_p)^a)$ are both finite of the same cardinality p^a .

By Cartier duality, it is equivalent to show that both $\text{Hom}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ and $\text{Ext}^1(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ have order p , and this is obvious (resolve the “first” $\mathbf{Z}/p\mathbf{Z}$ by

$$0 \rightarrow \mathbf{Z} \xrightarrow{p} \mathbf{Z} \rightarrow \mathbf{Z}/p\mathbf{Z} \rightarrow 0).$$

For another proof in this case, cf. Oort, [10], 85.

IV. ABSOLUTE FINITENESS THEOREMS

THEOREM 3. *Let \mathcal{O} be the ring of integers in a finite extension K of \mathbf{Q} . Let X be a smooth \mathcal{O} -scheme of finite type whose geometric generic fibre $X \otimes_{\mathcal{O}} \overline{K}$ is connected, and which maps surjectively to $\text{Spec}(\mathcal{O})$ (i.e. for every prime \mathfrak{p} of \mathcal{O} , the fibre over \mathfrak{p} , $X \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{p})$, is non empty). Then the group $\pi_1(X)^{ab}$ is finite.*

Proof. This follows immediately from Theorem 1 and global classfield theory, according to which $\pi_1(\text{Spec}(\mathcal{O}))^{ab}$, the galois group of the maximal unramified abelian extension of K , is finite. QED

THEOREM 4. *Let \mathcal{O} be the ring of integers in a finite extension K of \mathbf{Q} , $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ a finite set of primes of \mathcal{O} , $N = p_1 \dots p_n$ the product of their residue characteristics, and $\mathcal{O}[1/\mathfrak{p}_1 \dots \mathfrak{p}_n]$ the ring of “integers outside $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ ” in K . Let X be a smooth $\mathcal{O}[1/\mathfrak{p}_1 \dots \mathfrak{p}_n]$ -scheme of finite type, whose geometric generic fibre $X \otimes_{\mathcal{O}} \overline{K}$ is connected, and which maps surjectively to $\text{Spec}(\mathcal{O}[1/\mathfrak{p}_1 \dots \mathfrak{p}_n])$ (i.e. for every prime $\mathfrak{p} \notin \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$, the fibre*

$$X \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{p})$$

is non-empty). Then the group $\pi_1(X)^{ab}$ is the product of a finite group and a pro- N group.

Proof. Again an immediate consequence of Theorem 1 and global classfield theory, according to which $\pi_1(\text{Spec}(\mathcal{O}[1/\mathfrak{p}_1 \dots \mathfrak{p}_n]))^{ab}$, the galois group of the maximal abelian, unramified outside $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ -extension of K is finite times pro- N . QED