

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 27 (1981)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FINITENESS THEOREMS IN GEOMETRIC CLASSFIELD THEORY
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Kapitel: IV. Absolute Finiteness theorems
DOI: <https://doi.org/10.5169/seals-51753>

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By Cartier duality, it is equivalent to show that both $\text{Hom}(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ and $\text{Ext}^1(\mathbf{Z}/p\mathbf{Z}, \mathbf{Z}/p\mathbf{Z})$ have order p , and this is obvious (resolve the “first” $\mathbf{Z}/p\mathbf{Z}$ by

$$0 \rightarrow \mathbf{Z} \xrightarrow{p} \mathbf{Z} \rightarrow \mathbf{Z}/p\mathbf{Z} \rightarrow 0).$$

For another proof in this case, cf. Oort, [10], 85.

IV. ABSOLUTE FINITENESS THEOREMS

THEOREM 3. *Let \mathcal{O} be the ring of integers in a finite extension K of \mathbf{Q} . Let X be a smooth \mathcal{O} -scheme of finite type whose geometric generic fibre $X \otimes_{\mathcal{O}} \overline{K}$ is connected, and which maps surjectively to $\text{Spec}(\mathcal{O})$ (i.e. for every prime \mathfrak{p} of \mathcal{O} , the fibre over \mathfrak{p} , $X \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{p})$, is non empty). Then the group $\pi_1(X)^{ab}$ is finite.*

Proof. This follows immediately from Theorem 1 and global classfield theory, according to which $\pi_1(\text{Spec}(\mathcal{O}))^{ab}$, the galois group of the maximal unramified abelian extension of K , is finite. QED

THEOREM 4. *Let \mathcal{O} be the ring of integers in a finite extension K of \mathbf{Q} , $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ a finite set of primes of \mathcal{O} , $N = p_1 \dots p_n$ the product of their residue characteristics, and $\mathcal{O}[1/\mathfrak{p}_1 \dots \mathfrak{p}_n]$ the ring of “integers outside $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ ” in K . Let X be a smooth $\mathcal{O}[1/\mathfrak{p}_1 \dots \mathfrak{p}_n]$ -scheme of finite type, whose geometric generic fibre $X \otimes_{\mathcal{O}} \overline{K}$ is connected, and which maps surjectively to $\text{Spec}(\mathcal{O}[1/\mathfrak{p}_1 \dots \mathfrak{p}_n])$ (i.e. for every prime $\mathfrak{p} \notin \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$, the fibre*

$$X \otimes_{\mathcal{O}} (\mathcal{O}/\mathfrak{p})$$

is non-empty). Then the group $\pi_1(X)^{ab}$ is the product of a finite group and a pro- N group.

Proof. Again an immediate consequence of Theorem 1 and global classfield theory, according to which $\pi_1(\text{Spec}(\mathcal{O}[1/\mathfrak{p}_1 \dots \mathfrak{p}_n]))^{ab}$, the galois group of the maximal abelian, unramified outside $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ -extension of K is finite times pro- N . QED

THEOREM 5. *Let S be a normal, connected noetherian scheme, whose function field K is absolutely finitely generated. Let $f: X \rightarrow S$ be a smooth surjective morphism of finite type whose geometric generic fibre is connected, and which admits a cross-section $X \xrightarrow{\varepsilon} S$. Then there are only finitely many connected finite etale X -schemes Y/X which are galois over X with abelian galois group of order prime to $\text{char}(K)$ and which are completely decomposed over the marked section. If in addition we suppose X/S proper, we can drop the proviso "of order prime to $\text{char}(K)$ ".*

Proof. This is just the concatenation of Theorems 1 and 2 with the physical interpretation (1.3) of the group $\text{Ker}(X/S)$ in the presence of a section. QED

V. APPLICATION TO l -ADIC REPRESENTATIONS

Let l be a prime number, $\overline{\mathbf{Q}}_l$ an algebraic closure of \mathbf{Q}_l . By an l -adic representation ρ of a topological group π , we mean a finite-dimensional continuous representation

$$\rho: \pi \rightarrow GL(n, \overline{\mathbf{Q}}_l)$$

whose image lies in $GL(n, E_\lambda)$ for some finite extension E_λ of \mathbf{Q}_l .

THEOREM 6. (cf. Grothendieck, *via* [2], 1.3). *Let K be an absolutely finitely generated field, X/K a smooth, geometrically connected K -scheme of finite type, \bar{x} a geometric point of $X \otimes \overline{K}$, x the image geometric point of \bar{x} in X . Let l be a prime number, and ρ an l -adic representation of $\pi_1(X, x)$;*

$$\rho: \pi_1(X, x) \rightarrow GL(n, \overline{\mathbf{Q}}_l).$$

Let G be the Zariski closure of the image $\rho(\pi_1(X \otimes \overline{K}, \bar{x}))$ of the geometric fundamental group $\pi_1(X \otimes \overline{K}, \bar{x})$ in $GL(n, \overline{\mathbf{Q}}_l)$ and G^0 its identity component. Suppose that either l is different from the characteristic p of K , or that X/K is proper. Then:

- (1) *the radical of G^0 is unipotent, or equivalently:*
- (2) *if the restriction of ρ to the geometric fundamental group $\pi_1(X \otimes \overline{K}, \bar{x})$ is completely reducible, then the algebraic group G^0 is semi-simple.*