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ON COUSIN-I COMPLEX SPACES

by Edoardo BALLICO ¹⁾

SUMMARY

We study the properties of a complex space such that every closed analytic subspace has the 1-Cousin property. Under mild hypotheses we prove that X is a Stein space.

G. Berg in [2] proved very easily the following result: Let X be an open subset of a two-dimensional Stein space Y . If in X every 1-Cousin problem is solvable, then X is a domain of holomorphy in Y . It is not difficult to modify his proof and obtain that, under the stated assumption, X is always a Stein space.

In this note we want to consider this problem also for higher dimensional complex spaces. The proofs are always by induction on the dimension; they are similar to the proofs in the two dimensional case. This is also a partial generalization of my previous [1], theorem 2. We say that a complex space X is Cousin-I or has the 1-Cousin property if every 1-Cousin problem on X is solvable. Any Stein space is a Cousin-I space. We denote by $O(X)$ the algebra of holomorphic functions on a complex space X . For each $f \in O(X)$ we put

$$\{x \in X : f(x) = 0\} =: V(f),$$

an analytic subset of X . In this paper we consider for simplicity only complex spaces with bounded Zariski tangent dimension.

LEMMA 1. *Let X be a complex space with the 1-Cousin property and $f \in O(X)$ such that for each $x \in X$ the germ of f around x is a non-zero divisor in $\mathcal{O}_{X,x}$. We put*

$$Z := (V(f), \mathcal{O}_Z)$$

with

$$\mathcal{O}_Z := \mathcal{O}_X / f \mathcal{O}_{X|V(f)}.$$

Then the natural map $p: O(X) \rightarrow O(Z)$ is surjective.

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Proof: Let $\{U_i\}$ be a family of Stein open subsets of X which covers $V(f)$ and such that for each index i we have $U_i \cap V(f) \neq \emptyset$.

We consider the exact sequence of sheaves on X

$$0 \rightarrow \mathcal{O}_X \xrightarrow{t} \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0 \quad (1)$$

where t is the multiplication by f . It is exact because f is a non zero-divisor in $\mathcal{O}_{X,x}$ for each $x \in X$. We take $h \in \mathcal{O}(Z)$. From theorem B of Cartan-Serre it follows that for every i the restriction of h to $V(f) \cap U_i$ can be extended to a holomorphic function $f_i \in \mathcal{O}(U_i)$. If we have $U_i \cap U_j \neq \emptyset$, from the restriction of the exact sequence (1) to $U_i \cap U_j$, we obtain that $f_i - f_j = g_{ij} f$, where $g_{ij} \in \mathcal{O}(U_i \cap U_j)$ is uniquely determined because f is not a zero-divisor of $\mathcal{O}(U_i \cap U_j)$.

We take the covering of X given by $X \setminus V(f)$ and $\{U_i\}$. As data for the 1-Cousin problem in X we take the constant 1 on $X \setminus V(f)$ and the meromorphic function f_i/f on U_i . Again f_i/f is a meromorphic function on U_i , because f is not a zero-divisor on $\mathcal{O}_{X,x}$ for each $x \in U_i$.

We obtain a meromorphic function g on X which satisfies the thesis of the 1-Cousin problem. The function gf is holomorphic on X and gives an extension of h to X . \square

G. Berg in [2] lemma, gave a proof more or less of the lemma above, if $V(f)$ is a Stein space. He uses a strong theorem of Siu about the existence of a Stein open neighborhood of any Stein analytic subspace in any complex space. Our proof is similar, but does not use this strong theorem and is therefore more elementary.

THEOREM 1. *Let X be a complex n -dimensional space, such that $\mathcal{O}(X)$ gives local coordinates (in particular it is holomorphically spreadable). Suppose that X and every closed analytic subspace have the 1-Cousin property. Suppose that X is either reduced or Cohen-Macaulay. Then X is a Stein space.*

Proof: The proof is by induction on $n = \dim X$. If we have $n \leq 1$, then X is a Stein space because it does not contain any positive dimensional analytic compact subspace. Suppose $n \geq 2$.

We put as in [6] $S_d = \{x \in X : \text{prof}_x(\mathcal{O}_{X,x}) \leq d\}$. By [6] theorem 1.11, S_d is an analytic closed subspace of X with $\dim S_d \leq d$. For each $f \in \mathcal{O}(X)$, we have $\dim(V(f) \cap S_{k+1}) \leq k$ for each integer k if and only if f_x is a non zero-divisor in $\mathcal{O}_{X,x}$ for each $x \in X$ by [6], Corollary 1.18.

We may assume that X has no isolated point. Therefore, both under the hypotheses that X is reduced or that X is Cohen-Macaulay, we obtain $S_0 = \emptyset$.

By a simple application of Baire's category theorem, we obtain $f \in O(X)$ such that h has two different values on each irreducible component of dimension k of S_k .

First we want to prove that X is holomorphically separated. We fix two distinct points $x, y \in X$. If we have $f(x) \neq f(y)$, the result is trivially true. Suppose $f(x) = f(y)$. By considering instead of f the function $f - f(x)$, we may suppose that f vanishes at x . Therefore we suppose $x, y \in V(f)$ and we put $Z := (V(f), \mathcal{O}_Z)$ with

$$\mathcal{O}_Z := \mathcal{O}_X / f \mathcal{O}_{X|V(f)}.$$

a) If X is reduced, $V(f)$ with the reduced structure is by the inductive hypothesis a Stein space. But a complex space Y is a Stein space if and only if Y_{red} is a Stein space. Therefore Z is a Stein space, too.

b) If X is Cohen-Macaulay, then also Z is Cohen-Macaulay. In fact for each $x \in X$ with $f(x) = 0$, we have

$$\text{prof}(\mathcal{O}_{X,x} / f \mathcal{O}_{X,x}) = \text{prof}(\mathcal{O}_{X,x}) - 1$$

by [6], lemma 1.2, and

$$\dim(\mathcal{O}_{X,x} / f \mathcal{O}_{X,x}) = \dim \mathcal{O}_{X,x} - 1.$$

From the inductive hypothesis it follows that Z is a Stein space.

Therefore under both assumptions we have proved that Z is a Stein space. In particular there exists a $g \in O(Z)$ such that $g(x) \neq g(y)$. From lemma 1 it follows that there exists $G \in O(X)$ which extends g . In particular we have $G(x) \neq G(y)$, proving that X is holomorphically separated.

Let χ be a character of $O(X)$; χ is a multiplicative functional from $O(X)$ to \mathbb{C} . By [5], Corollary pag. 222, or [4], pag. 182, to prove that X is a Stein space it is sufficient to demonstrate that χ is a valuation in a point of X . We take $h \in O(X)$ as before, i.e. such that h has two different values on each irreducible component of dimension k of S_k . We put $f = h - \chi(h)$. We put $H := \ker(\chi)$. H is a maximal ideal of $O(X)$. The function f is in H . We define Z as above. Z is again a Stein space by the inductive hypothesis. From lemma 1 and the exact sequence (1) in lemma 1, we obtain an exact sequence

$$0 \rightarrow f O(X) \rightarrow O(X) \rightarrow O(Z) \rightarrow 0 \quad (2)$$

Since $f \in \ker(\chi)$, χ induces a character χ' on $O(Z)$. Since Z is a Stein space, χ' is induced by the valuation in a point $x \in Z \subset X$. If $\{g_i\} \in O(Z)$ generate $\ker(\chi')$

and $p(f_i) = g_i$, the functions f_i and f generate $H = \ker(\chi)$. Therefore, for every $m \in H$ we have $m(x) = 0$. Since M is a maximal ideal, χ is the valuation at the point x . \square

If $\dim X = 2$, then it is sufficient to assume the 1-Cousin property for X . In fact its 1-dimensional subspace Z in the proof above is a Stein space because it has no compact, analytic subspace of positive dimension. If X is Cohen-Macaulay, it follows from lemma 1 that it is sufficient to assume that X and every closed analytic subspace Z of X given by global equations f_1, \dots, f_k and with

$$\mathcal{O}_Z := \mathcal{O}_X / (f_1, \dots, f_k) \mathcal{O}_{X|Z}$$

are Cousin-1 space.

If in the theorem above we omit the condition that X has local coordinates given by global functions, we obtain only that every character is given by a valuation at a point of X . An easy inductive argument shows also that for every character χ of $\mathcal{O}(X)$, $\ker(\chi)$ is finitely generated. Therefore we obtain the following

PROPOSITION 1. *Let X be a holomorphically spreadable complex space such that every closed analytic subspace has the 1-Cousin property. Suppose that X is reduced or Cohen-Macaulay. Then*

- a) X is holomorphically separated,
- b) every character χ of $\mathcal{O}(X)$ is a valuation at a point of X and $\ker(\chi)$ is finitely generated.

THEOREM 2. *Let X be a complex reduced space which is a relatively compact open subset of a holomorphically separated, reduced complex space Y . Then X is a Stein space if and only if any closed analytic subspace, with its reduced structure, has the 1-Cousin property.*

Proof: The "only if" part is well-known. We have $\dim X = n < +\infty$ and we use induction on n . For $n \leq 1$ the result is well-known. Now suppose $n \geq 2$. Let $\{x_n\}$ be a sequence in X without accumulation points in X . We may suppose, eventually extracting a subsequence, that $\{x_n\}$ converges to a point $y \in \mathcal{O}(X)$. If we take into account the proof of [3], theorem 1.3, it remains only to prove that there exist

$$f_1, \dots, f_k \in \mathcal{O}(Y)$$

with only y as a common zero and such that there exist

$$g_1, \dots, g_k \in \mathcal{O}(X)$$

with $f_1g_1 + \dots + f_kg_k = 1$ on X . In fact one of the g_i 's must be unbounded on the sequence $\{x_n\}$. As in the proof of theorem 1 we take $f \in O(Y)$ a non zero-divisor in $\mathcal{O}_{Y,x}$ for each $x \in Y$ and vanishing in the point y . By the inductive hypothesis $V(f) \cap X$ with its reduced structure is a Stein space. We define $W := (V(f), \mathcal{O}_W)$ with

$$\mathcal{O}_W := \mathcal{O}_Y / f \mathcal{O}_{Y|V(f)}$$

and $Z := W|_{X \wedge W}$ is a Stein space, too. Let $p: O(X) \rightarrow O(Z)$ be the projection. From lemma 1 it follows that p is surjective. Let h_1, \dots, h_k be holomorphic functions on W such that y is their unique common zero. Since Z is a Stein space and h_1, \dots, h_k have no common zero in Z , there exist

$$g_1, \dots, g_k \in O(Z)$$

such that we have $g_1h_1|_Z + \dots + g_kh_k|_Z = 1$ on Z . We take $G_i \in O(X)$ such that $p(G_i)$ is g_i . By the exact sequence (2) we obtain that there exists $G \in O(X)$ such that we have $Gf + \dots + G_kh_k = 1$ on X . The proof is now finished. \square

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