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DESCARTES, EULER, POINCARÉ, PÓLYA—AND POLYHEDRA

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1. INTRODUCTION

When geometers talk of *polyhedra*, they restrict themselves to configurations, made up of *vertices*, *edges* and *faces*, embedded in three-dimensional Euclidean space. Indeed, their polyhedra are always homeomorphic to the two-dimensional sphere S^2. Here we adopt the topologists' terminology, wherein dimension is a topological invariant, intrinsic to the configuration, and *not* a property of the ambient space in which the configuration is located. Thus S^2 is the surface of the 3-dimensional ball; and so we find, among the geometers' polyhedra, the five Platonic "solids", together with many other examples. However, we should emphasize that we do not here think of a Platonic "solid" as a solid; we have in mind the bounding surface of the solid, not the interior. It seems to us that geometers are sometimes able to be cavalier about this distinction (so that, for them, a *polygon* may be the closed polygonal path or the homeomorph of a disk), but we will need, in what follows, to be precise about meanings.

In this article we retrace an interesting historical path in the study of polyhedra and even carry the story further ourselves—though with modest expectations! We begin with a result due to Descartes (1596-1650). Let us consider a convex polyhedron P, homeomorphic to S^2. Euclid proved that the sum of the face angles at any vertex P is less than 2π; the difference between this sum and 2π is called the *angular defect* at that vertex. If we sum the angular defects over all the vertices of P we obtain the total angular defect Δ; Descartes proved, using methods of spherical trigonometry, that $\Delta = 4\pi$ for every convex polyhedron P. Thus in Figure 1 (b) there are 8 identical vertices on the cube and the angular defect at every vertex is $\frac{\pi}{2}$, so that the total angular defect Δ is 4π. Notice that the polyhedra shown in Figure 2 are not homeomorphic to S^2 and they fail to satisfy the formula.

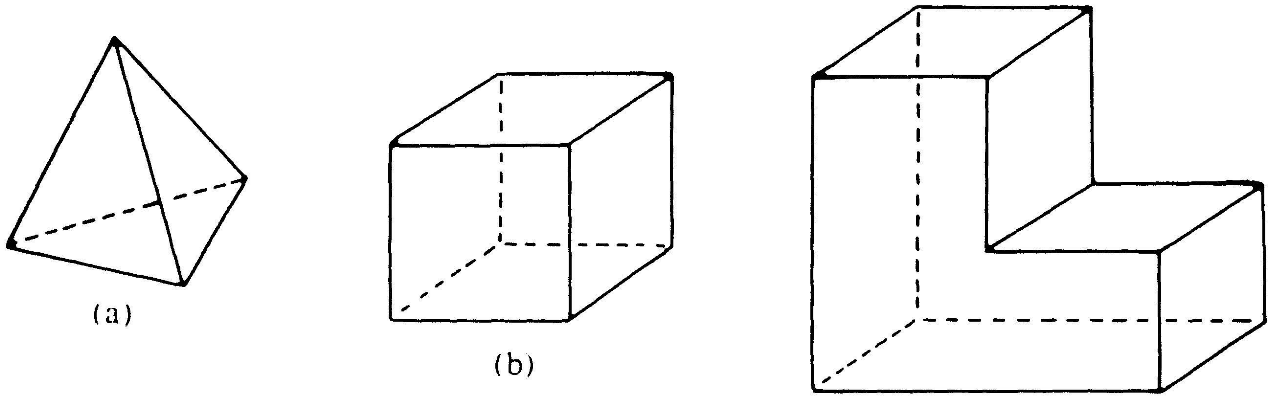

FIGURE 1

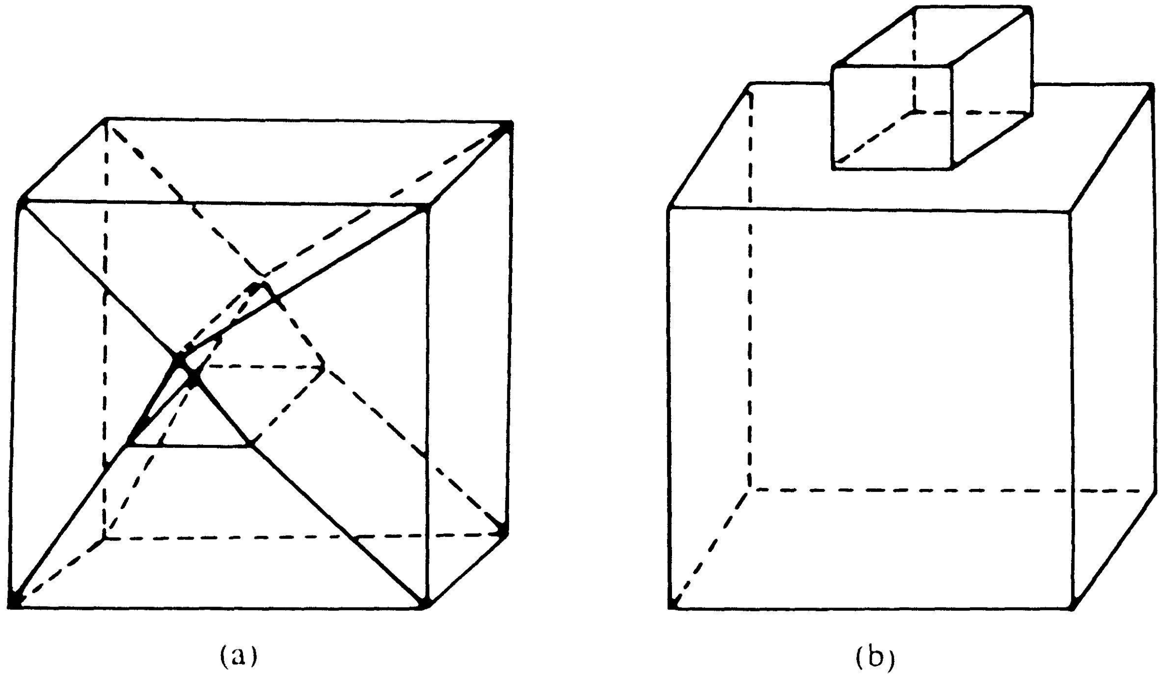

FIGURE 2

Pólya gave an argument in a lecture at Stanford University on March 6, 1974 (see [1]) to deduce Descartes' theorem, using the fact that the Euler characteristic of any polyhedron homeomorphic to S^2 is 2. Here the Euler characteristic $\chi(P)$ is given by the formula

$$\chi(P) = V - E + F, \tag{1.1}$$

where V is the number of vertices of P, E is the number of edges of P, and F is the number of faces of P. Thus Pólya's proof (which appears in slightly modified form in [2]) shows that $\Delta = 2\pi\chi$ and hence $\Delta = 4\pi$ since $\chi(P) = 2$ when P is homeomorphic to S^2.

However Pólya's proof really demonstrates a much more general fact; namely that

$$\Delta = 2\pi\chi \tag{1.2}$$

for *any* 2-dimensional polyhedral manifold [1]). Thus if S is any rectilinear surface, subdivided into vertices, edges and faces in such a way that every edge is incident with exactly two faces, then formula (1.2) holds for S. Of course, we have to interpret Δ somewhat more generally in the sense that, since we no longer require convexity, we must allow the angular defect at any vertex to be negative.

Let us now take S to be any closed surface, orientable or not. Then we may find a homeomorphic rectilinear model T of S, and we may compute $\Delta(T)$, $\chi(T)$. Since we know that $\chi(T)$ is a *topological* invariant of S—a result due to Poincaré—it follows that $\Delta(T)$, too, is a topological invariant of S, a result which is surely rather surprising.

In the next section we give, in its more general setting, Pólya's proof of the relation (1.2), and point to the topological significance of the result. In Section 3 we consider analogous formulae for $\Delta(P)$, where P is a polyhedron of dimension greater than 2. Now Schläfli [9] generalized Euler's formula to spheres of higher dimension. He succeeded in demonstrating that if P is a polyhedral subdivision of the n-dimensional sphere S^n and if N_i is the number of i-dimensional cells in the subdivision, then

$$\begin{aligned} \chi(P) &= 2 \qquad \text{if } n \text{ is even,} \\ \chi(P) &= 0 \qquad \text{if } n \text{ is odd,} \end{aligned} \tag{1.3}$$

where

$$\chi(P) = \sum_{i=0}^{n} (-1)^i N_i, \tag{1.4}$$

We call this alternating sum (1.4) the *Euler-Poincaré characteristic* of P and note that it may be defined for any polyhedron P, of any dimension. Poincaré [10] proved that $\chi(P)$ is a topological invariant. This means that if X is any geometric configuration embedded in some Euclidean space (of arbitrary dimension) and if P, Q are any two polyhedra, subdivided into cells of dimension $0, 1, 2, \ldots, n$ (vertices, edges, faces, ...), such that P and Q are each homeomorphic to X, then $\chi(P) = \chi(Q)$. This result is one of the great triumphs of homology

[1]) Here, of course, we use the term "polyhedron" in the more general sense favored by topologists. Thus a polyhedron, in this broader sense, certainly need not be 2-dimensional; and an n-dimensional polyhedron need not be homeomorphic to an n-dimensional sphere.

theory [12, p. 167]. For there are natural numbers $p_0, p_1, ..., p_n$ measuring the number of "holes" in X of dimensions 0, 1, ..., n, and one may show that, for any polyhedron P homeomorphic to X,

$$\Sigma(-1)^i N_i = \Sigma(-1)^i p_i . \tag{1.5}$$

The numbers $p_0, p_1, ..., p_n$ are called the *Betti numbers* of X; they are the dimensions of the homology groups of X in dimensions 0, 1, ..., n. For an n-dimensional sphere S^n, we have

$$p_0(S^n) = p_n(S^n) = 1, p_i(S^n) = 0, i \neq 0, n ; \tag{1.6}$$

thus (1.5) and (1.6) explain Schläfli's result (1.3).

For any polyhedron P, we may continue to define the total angular defect $\Delta(P)$ exactly as in the two-dimensional case. However, $\Delta(P)$ obviously depends only on the two-dimensional structure of P—its vertices, edges and faces—so that we cannot expect, for higher-dimensional polyhedra, either that $\Delta(P)$ will be an invariant or that it will be related to the Euler-Poincaré characteristic. However, we may still attempt to generalize Pólya's argument and thus to express $\Delta(P)$ as a function of V, E and F (or, in our present notation, N_0, N_1 and N_2).

We prove in Section 3 that, indeed, $\Delta(P)$ may be expressed in terms of N_0, N_1, N_2, provided only that the cellular structure on P has the property that there exists an integer q such that every edge of P is incident with exactly q faces. We give three examples of standard cellular subdivisions of S^n with this property. Reverting to the language of geometers (as exemplified by Coxeter [3]), such structures on S^n are called *polytopes*, and the three polytopes considered are called, in [3], respectively *simplexes*, *cross polytopes*, and *parallelotopes*—and will be so referred to by us. The numbers q in these cases are, respectively n, $2n - 2$, n. We compute Δ in these three cases. We remark that the fact that Δ is, in these cases, a function of N_0, N_1, and N_2 shows that it is a combinatorial, rather than a geometric, invariant; that is, we may pull and push the n-sphere around, squeeze it, squash it, elongate it, stretch it, without altering Δ. Once again our intuition may be at fault!

We close this article with a brief resumé of the history of the question. In this resumé, as in the article itself, we do not take account of another direction in which it may be said that formula (1.2) has been generalized—in the direction of differential geometry. For formula (1.2) contains the seeds of the celebrated

Gauss-Bonnet formula for smooth manifolds; an excellent account of the development in this direction is to be found in the article by Chern ([13]; see especially formula (4) on p. 343).

2. PÓLYA'S PROOF OF DESCARTES' THEOREM

We start from the position that Euler's formula for a polyhedral 2-sphere S^2 is known; that is to say, if P is a polyhedron homeomorphic to S^2 with V vertices, E edges and F faces, then

$$V - E + F = 2. \tag{2.1}$$

In Figure 1 (a), for example, $V = 4$, $E = 6$, $F = 4$. Thus $4 - 6 + 4 = 2$, verifying (2.1). Euler's formula is discussed in many elementary books on polyhedra and many proofs have been given. The book by Courant and Robbins, *What is Mathematics?* [4] contains a proof using networks. Pólya's book, *Mathematics and Plausible Reasoning*, Vol. I, [1], has a sequence of problems that leads the reader to a proof. Lakatos' *Proofs and Refutations* [8] is cleverly written in the format of a dialogue between a mathematics teacher and his extremely bright students (who continually find counterexamples to the proposed theorems). The "general" proof must be attributed to Poincaré [10] who, as explained in the Introduction, proved that the generalized Euler-Poincaré characteristic is a topological invariant which takes the value 2 on any even-dimensional sphere.

We now show how Pólya deduced Descartes theorem from (2.1); this argument is essentially that given in [2].

Let P be a polyhedron homeomorphic to S^2, subdivided into vertices, edges and faces in such a way that every edge is incident with exactly two faces. Number the vertices $1, 2, \ldots, V$ and let the sum of the plane face angles at the i-th vertex be σ_i. Then the angular defect at the i-th vertex is

$$\delta_i = 2\pi - \sigma_i .$$

Note that δ_i will be positive if P is convex, but that, in general, δ_i may be negative or zero. Let

$$\Delta = \sum_{i=1}^{V} \delta_i .$$

We want to show that $\Delta = 4\pi$.