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THEOREM 2. Let  $q$  be an even positive integer. Then :

$$|S(a_1, \dots, a_k; q)| \leq 2^{\frac{k+1}{2}} k^{v(q)} q^{\frac{k-1}{2}} (a_1, a_k, q)^{\frac{1}{2}} \dots (a_{k-1}, a_k, q)^{\frac{1}{2}}.$$

Estermann, [2], has dealt with the case of the Kloosterman sum.

## 2. LEMMAS

*Lemma 1.* Consider the congruence:

$$x^k \equiv a \pmod{p^m}$$

where  $k, m$  are positive integers,  $a$  is an integer,  $p$  a prime and  $(a, p) = 1$ . Then:

1. If  $p > 2$ , this congruence has at most  $k$  incongruent solutions mod  $p^m$ .
2. If  $p = 2$  and  $k$  is odd, then this congruence has exactly 1 solution mod  $p^m$ .
3. If  $p = 2$ , and  $k = 2^r l$ ,  $r > 1$ ,  $l$  odd, then this congruence has at most  $\min\{2^{r+1}, p^m\}$  solutions mod  $p^m$ .

*Proof:* This is essentially found on pp. 115, 119 of [3].

*Lemma 2.* Let  $p$  be a prime, and  $m, n$  positive integers,  $\frac{1}{2}m \leq n < m$ . Let  $y_1, \dots, y_{k-1}, z_1, \dots, z_{k-1}$  be integers;  $p \nmid y_1, \dots, p \nmid y_{k-1}$ . Define  $[y_1, \dots, y_{k-1}; p^m]$  as that integer  $y$ ,  $0 < y < p^m$  such that  $y(y_1 \dots y_{k-1}) \equiv 1 \pmod{p^m}$ . Then:

$$\begin{aligned} [y_1 + p^n z_1, \dots, y_{k-1} + p^n z_{k-1}; p^m] &\equiv [y_1, \dots, y_{k-1}; p^m] \\ &\quad - [y_1; p^m]^2 [y_2; p^m] \dots [y_{k-1}; p^m] p^n z_1 \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &\quad - [y_1; p^m] \dots [y_{k-2}; p^m] [y_{k-1}; p^m]^2 p^n z_{k-1} \pmod{p^m} \end{aligned}$$

*Proof:* This follows from the relation

$$[y_1; p^m] \dots [y_{k-1}; p^m] \equiv [y_1, \dots, y_{k-1}; p^m] \pmod{p^m}$$

and Lemma 1 of [2].

*Lemma 3.* Let  $p$  be a prime,  $m, n$  positive integers,  $m = 2n + 1$ . Let  $y_1, \dots, y_{k-1}, z_1, \dots, z_{k-1}$  be integers;  $p \nmid y_1, \dots, p \nmid y_{k-1}$ . Then

$$\begin{aligned}
 & [y_1 + p^n z_1, \dots, y_{k-1} + p^n z_{k-1}; p^m] \equiv [y_1, \dots, y_{k-1}; p^m] \\
 & + [y_1; p^m]^3 [y_2; p^m] \dots [y_{k-1}; p^m] p^{2n} z_1^2 \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot \\
 & + [y_1; p^m] \dots [y_{k-2}; p^m] [y_{k-1}; p^m]^3 p^{2n} z_{k-1}^2 \\
 & - [y_1; p^m]^2 [y_2; p^m] \dots [y_{k-1}; p^m] p^n z_1 \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot \\
 & - [y_1; p^m] \dots [y_{k-2}; p^m] [y_{k-1}; p^m]^2 p^n z_{k-1} \\
 & + [y_1; p^m]^2 [y_2; p^m]^2 [y_3; p^m] \dots [y_{k-1}; p^m] p^{2n} z_1 z_2 \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot \\
 & + [y_1; p^m]^2 [y_2; p^m] \dots [y_{k-2}; p^m] [y_{k-1}; p^m]^2 p^{2n} z_1 z_{k-1} \\
 & + [y_1; p^m] [y_2; p^m]^2 [y_3; p^m]^2 [y_4; p^m] \dots [y_{k-1}; p^m] p^{2n} z_2 z_3 \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot \\
 & + [y_1; p^m] [y_2; p^m]^2 [y_3; p^m] \dots [y_{k-2}; p^m] [y_{k-1}; p^m]^2 p^{2n} z_2 z_{k-1} \\
 & \quad \cdot \\
 & \quad \cdot \\
 & \quad \cdot \\
 & + [y_1; p^m] \dots [y_{k-3}; p^m] [y_{k-2}; p^m]^2 [y_{k-1}; p^m]^2 p^{2n} z_{k-2} z_{k-1} \\
 & \pmod{p^m}
 \end{aligned}$$

*Proof:* This follows from Lemma 5 of [2].

*Lemma 4.* Let  $p > 2$  be a prime, and  $n$  a positive integer. Let  $a, h$  be integers. Then:

$$\left| \sum_{0 \leq z < p^{n+1}} e(az^2 p^{-1} + hzp^{-n-1}) \right| = \begin{cases} 0 & p^n \nmid h \\ p^{n+\frac{1}{2}} & p^n \mid h, \quad p \nmid a \\ p^{n+1} & p^{n+1} \mid h, \quad p \mid a \\ 0 & p^{n+1} \nmid h, \quad p \mid a. \end{cases}$$

*Proof:* The first two parts of this lemma are Lemma 5 of [2]. The last two parts are trivial.

### 3. PROOF OF THEOREMS 1 AND 2

PROPOSITION 1. Let  $p$  be a prime,  $m$  a positive integer and  $a_1, \dots, a_k$ , integers such that

$$(a_1, a_k, p^m) = \dots = (a_{k-1}, a_k, p^m) = p^h \quad 0 \leq h < m.$$

Then

$$S(a_1, \dots, a_k; p^m) = (p^h)^{k-1} S(a_1 p^{-h}, \dots, a_k p^{-h}; p^{m-h})$$

*Proof:* The proof is similar to that of [2], page 85 bottom.

PROPOSITION 2. Let  $m, n$  be positive integers  $\frac{1}{2}m \leq n < m$ ,  $p$  a prime, and  $a_1, \dots, a_k$  integers such that  $(a_1, a_k; p^m) = 1$ . Then:

$$|S(a_1, \dots, a_k; p^m)| \leq A(p^n)^{k-1}$$

where

$$A = \begin{cases} k & \text{if } p > 2. \\ 1 & \text{if } p = 2 \text{ and } k \text{ is odd.} \\ \min \{ 2^{r+1}, p^m \} & \text{if } p = 2 \text{ and } k = 2^r l, \\ & r > 1 \text{ and } l \text{ odd.} \end{cases}$$

*Proof:* Let us assume throughout this proposition that  $S(a_1, \dots, a_k; p^m) \neq 0$ , or else we are done.

Now we have the identity