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This result rapidly yields our version of the "basis theorem" of the Schubert calculus, namely

THEOREM 2.9. Ker $(c) = I_W$ *and* c *induces an isomorphism* $S_W \approx H_W$.

Proof. For the first assertion, by (2.8), it suffices to compute

$$c(\lambda d) = \lambda \sum \varepsilon \Delta_w(d) X_w = \lambda \Delta_{w_0}(d) X_{w_0}$$
$$= \lambda \,|\, W \,|\, X_{w_0} .$$

Finally, c is clearly onto by construction.

In the next section we will work on producing an explicit section for c.

Remark. Demazure's proof, though restricted to Weyl groups, is done integrally. In that situation, c is not onto, and Demazure computes the order of the finite quotient. It corresponds to the usual notion of torsion in Lie groups [3, 5]. Indeed, the point is that only when W preserves some integral lattice can one hope to carry out an analysis in integral cohomology; in the general case we must resort to real cohomology, as we do here. Of course, the torsion problems then disappear.

3. GIAMBELLI FORMULA

We begin with an easy lemma.

LEMMA 3.1. Δ *is quasi-multiplicative, i.e.*

$$\Delta_w \cdot \Delta_{w'} = \begin{cases} \Delta_{ww'} & \text{if } l(ww') = l(w) + l(w') \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first clause is immediate since the conditions implies that reduced decompositions of w and w' can be juxtaposed to yield a reduced decomposition of ww'. Now suppose $w = s_\alpha w'$ and $l(s_\alpha w') = l(w') - 1$ (that this is the only possibility that follows from (1.1)). Then $w' = s_\alpha(s_\alpha w')$ and

$$l(w') = 1 + \big(l(w') - 1\big) = l(s_\alpha) + l(s_\alpha w')$$

so by the first part $\Delta_{w'} = \Delta_{s_\alpha} \Delta_{s_\alpha w'}$. But

$$0 = \Delta_{s_\alpha} \Delta_{s_\alpha} \Delta_{s_\alpha w'} = \Delta_{s_\alpha} \Delta_{w'}$$

by (2.2 ii) and induction on $l(w)$ completes the proof.

COROLLARY 3.2. $\varepsilon \cdot \Delta_{w'} \Delta_{w^{-1}w_0} = \delta_{ww'} \Delta_{w_0}$ *on* $S_N(V)$.

Proof. If $w' = w$, then by (1.4) and (3.1)

$$\Delta_w \Delta_{w^{-1}w_0} = \Delta_{w_0}$$

and the result follows.

We now need only consider $w' \neq w$, but with $l(w) = l(w')$, (otherwise, we are done for dimensional reasons). Thus

$$l(w') + l(w^{-1}w_0) = l(w') + (l(w_0) - l(w)) = l(w_0)$$

and

$$l(w'w^{-1}w_0) = l(w_0) - l(w'w^{-1}) \neq l(w_0)$$

So by (3.1), $\Delta_{w'} \Delta_{w^{-1}w_0} = 0$, and the proof is complete.

It is now easy to dualize this to the following assertion:

COROLLARY 3.3 (Giambelli formula). $c\left(\Delta_{w^{-1}w_0}\left(\frac{d}{|W|}\right)\right) = X_w$. *Hence in particular*, $c\left(\frac{d}{|W|}\right) = X_{w_0}$.

Proof.
$$c\left(\Delta_{w^{-1}w_0}\left(\frac{d}{|W|}\right)\right) = \sum_{w' \in W} \varepsilon \Delta_{w'}\left(\Delta_{w^{-1}w_0}\left(\frac{d}{|W|}\right)\right) X_{w'}$$
$$= \sum_{\substack{w' \in W \\ l(w') = l(w)}} \delta_{ww'}\, \varepsilon \Delta_{w_0}\left(\frac{d}{|W|}\right) X_{w'}$$
$$= X_w \qquad \text{by (2.5).}$$

Note that the map $\sigma : X_w \mapsto \Delta_{w^{-1}w_0}\left(\frac{d}{|W|}\right)$ is a vector space section for c. In the remainder of this section we will find other I_W-equivalent expressions for X_{w_0} and use these to put σ in a more manageable form. We will call X_{w_0} the *fundamental class* of the cohomology ring H_W.

Example. Let $W = W(A_{n-1}) = \Sigma_n$. As usual, the positive roots Δ^+ are $\{e_i - e_j : i < j\}$ where $\{e_i\}$ is the standard basis of $\mathbf{R}^n$. Hence, the fundamental class is c of a multiple of the Vandermonde determinant, namely

$$\frac{1}{n!}\begin{vmatrix} 1 & e_n & e_n^2 & \cdots & e_n^{n-1} \\ 1 & e_{n-1} & e_{n-1}^2 & \cdots & e_{n-1}^{n-1} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & e_1 & e_1^2 & \cdots & e_1^{n-1} \end{vmatrix}$$

In this example we used the standard basis for V. The following result indicates that a Coxeter generalization of the fundamental weight basis is more appropriate in our situation. Recall the *fundamental weights* $\{\omega_\alpha\}_{\alpha\in\Sigma}$ are given by the requirement

$$(\omega_\alpha, \beta^v) = \delta_{\alpha\beta}$$

We now have

LEMMA 3.4.

(i) $\Delta_\beta(\omega_\alpha) = \delta_{\alpha\beta}$,

(ii) $c(\omega_\alpha) = X_{s_\alpha}$,

(iii) $c(\alpha) = \sum_{\beta\in\Sigma} (\alpha, \beta^v) X_{s_\beta}$.

Proof.

(i) $\Delta_\beta(\omega_\alpha) = \beta^{-1}(\omega_\alpha - s_\beta(\omega_\alpha)) = \beta^{-1}(\omega_\alpha - (\omega_\alpha - (\omega_\alpha, \beta^v)\beta))$
$= (\omega_\alpha, \beta^v) = \delta_{\alpha\beta}$

(ii) $c(\omega_\alpha) = \sum_{w\in W} \varepsilon \Delta_w(\omega_\alpha) X_w = \sum_{\beta\in\Sigma} \Delta_\beta(\omega_\alpha) X_{s_\beta} = X_{s_\alpha}$

(iii) Since $\alpha = \sum_{\beta\in\Sigma} (\alpha, \beta^v)\,\omega_\beta$, the result follows immediately from (ii).

This result tells us that if we can write X_w as c of some polynomial in the $\{\omega_\alpha\}_{\alpha\in Z}$ or $\{\alpha\}_{\alpha\in Z}$ we will have also written X_w as a polynomial in the X_{s_α}'s. We will often abbreviate the Cartan matrix entries by $c_{\alpha,\beta} = (\alpha, \beta^v)$ $= -\dfrac{\|\alpha\|}{\|\beta\|}\cos\left(\dfrac{\pi}{m_{\alpha\beta}}\right)$. In practice, it is maximally efficient to write X_w as a polynomial in the simple roots, since then an easy substitution will yield either a polynomial in the weights or a polynomial in the original coordinate variables $e_1, \ldots, e_n$.

It is possible to relate the fundamental class X_{w_0}, with the invariant theory of W.

PROPOSITION 3.5. *Let* $f_1, \ldots, f_n$ *be fundamental invariants for* W. *Then, if* $J = \det\left(\dfrac{\partial f_i}{\partial x_j}\right)$ *is the Jacobian of these polynomials there is a real number* λ *such that*

$$c(\lambda J) = X_{w_0}.$$

Proof. This follows from the stronger, well-known assertion that d divides J [20, p. 85]. (It also follows from the theory of complete intersection rings.)

In the interest of understanding the Giambelli formula (3.3) we deduce some formulae for $\Delta_w(d)$. If $\{\alpha_i\}_{1 \leq i \leq n}$ are distinct positive roots we denote by $d_{\alpha_1, \ldots, \alpha_n}$ the product $d \cdot \prod_{i=1}^{n} \alpha_i^{-1} = \prod_{\substack{\alpha \in \Delta \\ \alpha \neq \alpha_i}} \alpha$. It is easy to see

LEMMA 3.6.

$$s_\beta(d_{\alpha_1, \ldots, \alpha_n}) = \begin{cases} d_{s_\beta(\alpha_1), \ldots, s_\beta(\alpha_n), \beta} & \text{if } \beta = \alpha_j, \\ -d_{s_\beta(\alpha_1), \ldots, s(\alpha_n)} & \text{otherwise.} \end{cases}$$

Proof. Since s_β permutes the set $\Delta^+ - \{\beta\}$, it also permutes

$$\Delta^+ - \{\beta, \alpha_1, \ldots, \alpha_n, s_\beta(\alpha_1), \ldots, s_\beta(\alpha_n)\},$$

where $\beta \neq \alpha_i$, for all i. Hence

$$\begin{aligned} s_\beta(d_{\alpha_1, \ldots, \alpha_n}) &= s_\beta(d_{\beta, \alpha_1, \ldots, \alpha_n, s_\beta(\alpha_1), \ldots, s_\beta(\alpha_n)}) \cdot s_\beta(\beta) \cdot s_\beta^2(\alpha_1) \circ \ldots \circ s_\beta^2(\alpha_n) \\ &= d_{\beta, \alpha_1, \ldots, \alpha_n, s_\beta(\alpha_1), \ldots, s_\beta(\alpha_n)} \cdot (-\beta) \cdot \alpha_1 \cdot \ldots \cdot \alpha_n \\ &= -d_{s_\beta(\alpha_1), \ldots, s_\beta(\alpha_n)} \end{aligned}$$

Similarly in the other case.

PROPOSITION 3.7.

$$\Delta_\beta(d_{\alpha_1, \ldots, \alpha_n}) = \begin{cases} \displaystyle\sum_{\substack{s \neq \phi \\ s \subseteq \{1, \ldots, \hat{j}, \ldots, n\}}} (-1)^{|s|} \prod_{i \in s} c_{\alpha_i, \beta} \cdot \\ \beta^{|s|-1} d_{\{\alpha_i : i \in s\}, s_\beta(\alpha_1), \ldots, \widehat{s_\beta(\alpha_j)}, \ldots, s_\beta(\alpha_n)} \\ \text{if } \beta = \alpha_j \\ d_{\alpha_1, \ldots, \alpha_n, \beta} + d_{s_\beta(\alpha_1), \ldots, s_\beta(\alpha_n), \beta} \\ \text{otherwise} \end{cases}$$

Proof. The second case is easy so we look at the first

$$\begin{aligned} \Delta_\beta(d_{\alpha_1, \ldots, \alpha_n}) &= \beta^{-1}(d_{\alpha_1, \ldots, \alpha_n} - d_{s_\beta(\alpha_1), \ldots, \widehat{s_\beta(\alpha_j)}, \ldots, s_\beta(\alpha_n), \beta}) \\ &= \beta^{-1}[d_{\alpha_1, \ldots, \alpha_n, s_\beta(\alpha_1), \ldots, \widehat{s_\beta(\alpha_j)}, \ldots, s_\beta(\alpha_n)} \\ &\quad \cdot (s_\beta(\alpha_1) \cdot \ldots \cdot \widehat{s_\beta(\alpha_j)} \cdot \ldots \cdot s_\beta(\alpha_n) - \alpha_1 \cdot \ldots \cdot \hat{\alpha}_j \cdot \ldots \cdot \alpha_n)] \\ &= d_{\alpha_1, \ldots, \alpha_n, s_\beta(\alpha_j), \ldots, \widehat{s_\beta(\alpha_j)}, \ldots, s_{\beta n}(\alpha)} \\ &\quad \beta^{-1}(\prod_{i \neq j} (\alpha_i - (\alpha_i, \beta^v)\beta) - \prod_{i \neq j} \alpha_i) \end{aligned}$$

and after writing the product as a sum the desired expression follows.

It is possible to use (3.7) to explicitly compute polynomial expressions for X_w.

Example. Let $W = W(A_2)$ where A_2 is the root system in $\mathbf{R}^3$ with simple roots $\Sigma = \{\alpha = e_1 - e_2, \beta = e_2 - e_3\}$ and the additional positive root $\alpha + \beta = e_1 - e_3$. Hence $X_{w_0} = \frac{1}{6}\alpha\beta(\alpha+\beta)$. As a check of this we compute the Jacobian J of the fundamental invariant. Recall

$$\sigma_1 = -(e_2+e_3)(e_2+e_3) + e_2 e_3$$

and

$$\sigma_2 = -(e_2+e_3)e_2 e_3,$$

where we have eliminated $e_1 = -(e_2+e_3)$. Then:

$$J = 3(e_2^2 e_3 - e_3^2 e_2) + 2(e_2^3 - e_3^3) = d,$$

so also, $X_{w_0} = \frac{1}{6} J$. Now by (3.7) we can compute

$$\Delta_\alpha\left(\frac{d}{6}\right) = \tfrac{1}{6}(2d_\alpha) = \tfrac{1}{3}\beta(\alpha+\beta)$$

and

$$\Delta_\beta \Delta_\alpha\left(\frac{d}{6}\right) = \tfrac{1}{3}(\Delta_\beta d_\alpha)) = \tfrac{1}{3}(d_{\alpha,\beta} + d_{s_\beta(\alpha),\beta}) = \tfrac{1}{3}(\alpha+\beta+\alpha) = \tfrac{1}{3}(2\alpha+\beta)$$

so that:

$$X_{s_\alpha s_\beta} = \tfrac{1}{3}\beta(\alpha+\beta) \quad \text{and} \quad X_{s_\alpha} = \tfrac{1}{3}(2\alpha+\beta) = \omega_\alpha$$

as one easily checks.

Now since the Cartan matrix is $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ we have

$$\alpha = 2\omega_\alpha - \omega_\beta$$
$$\beta = -\omega_\alpha + 2\omega_\beta$$

so for example

$$X_{s_\alpha s_\alpha} = \tfrac{1}{3}(-X_{s_\alpha} + 2X_{s_\alpha})(X_{s_\alpha} + X_{s_\beta})$$
$$= \tfrac{1}{3}(-X_{s_\alpha}^2 + X_{s_\beta}X_{s_\alpha} + 2X_{s_\beta}^2)$$

which will be confirmed further in the next section.

Remark. In the crystallographic case, it follows from the Weyl denominator formula (see [6, p. 185], [2, p. 17]) that

$$\frac{d}{|W|} \equiv \frac{\rho^N}{N!} \pmod{I_W}$$

where ρ is the sum of the fundamental weights. Hence one can attempt to compute the operators Δ_w on ρ^N.

It is possible to develop such formulae and we hope to treat them elsewhere. In particular, one might want to conjecture in the general case that $\rho^N \notin I_W$, maybe even for all ρ in the interior of the fundamental chamber.

4. Pieri formula

Recall that the algebra of operators $\mathbb{\Delta}_W$ was generated by both the Δ_α's and the multiplication operators ω^*. Using the basis constructed in (2.9), if one composes such operators, say $\omega^* \circ \Delta_w$ or $\Delta_w \circ \omega^*$, it should be possible to express them linearly in terms of the operators Δ_g, $g \in W$. Of course, our eventual concern is with the algebra $\mathbb{\Delta}_W$ and

$$\varepsilon \circ \omega^* \cdot \Delta_w = 0 \,.$$

So, if we compute the commutator $[\Delta_w, \omega^*]$ a quick application of ε will yield a formula for $\varepsilon \cdot \Delta_w \circ \omega^*$. Here we are following the strategy of Bernstein-Gelfand-Gelfand [2]. Essentially, this result is our Pieri formula disguised in its dual form.

In order for the techniques of section 1 and induction to be easily applicable, we work with the slightly modified operator $w^{-1}\Delta_w$ (recall $W \subset \mathbb{\Delta}_W$). The main result is

THEOREM 4.1. *If* $w \in W, \omega \in V^*$, *then in* End $S(V)$,

$$[w^{-1}\Delta_w, \omega^*] = \sum_{w' \xrightarrow{\gamma} w} \left(w'^{-1}(\gamma)^v, \omega\right) w^{-1}\Delta_{w'} \,.$$

We will now fix a reduced decomposition $w = s_{\alpha_1}, \ldots, s_{\alpha_k}$ and write s_i for s_{α_i} and $w_i = s_{\alpha_n} \ldots s_{\alpha_i}$. First we have the following easy observation.

LEMMA 4.2. *Let* $\theta_i = s_k \ldots s_{i+1}(\alpha_i) = w_{i+1}(\alpha_i)$, $1 \leqslant i \leqslant k$. *Then*

(i) $w^{-1}\Delta_w = \Delta_{\theta_1}\Delta_{\theta_2} \cdots \Delta_{\theta_k}$

and

(ii) $s_{\theta_i}(w\hat{_i})^{-1} = w^{-1}$

Proof. Note by (2.2 ii, iv) $s_\alpha \Delta_\alpha = \Delta_\alpha$. Hence