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$$\frac{d}{|W|} \equiv \frac{\rho^N}{N!} \pmod{I_W}$$

where  $\rho$  is the sum of the fundamental weights. Hence one can attempt to compute the operators  $\Delta_w$  on  $\rho^N$ .

It is possible to develop such formulae and we hope to treat them elsewhere. In particular, one might want to conjecture in the general case that  $\rho^N \notin I_W$ , maybe even for all  $\rho$  in the interior of the fundamental chamber.

#### 4. PIERI FORMULA

Recall that the algebra of operators  $\Delta_W$  was generated by both the  $\Delta_\alpha$ 's and the multiplication operators  $\omega^*$ . Using the basis constructed in (2.9), if one composes such operators, say  $\omega^* \circ \Delta_w$  or  $\Delta_w \circ \omega^*$ , it should be possible to express them linearly in terms of the operators  $\Delta_g$ ,  $g \in W$ . Of course, our eventual concern is with the algebra  $\Delta_W$  and

$$\varepsilon \circ \omega^* \cdot \Delta_w = 0 .$$

So, if we compute the commutator  $[\Delta_w, \omega^*]$  a quick application of  $\varepsilon$  will yield a formula for  $\varepsilon \cdot \Delta_w \circ \omega^*$ . Here we are following the strategy of Bernstein-Gelfand-Gelfand [2]. Essentially, this result is our Pieri formula disguised in its dual form.

In order for the techniques of section 1 and induction to be easily applicable, we work with the slightly modified operator  $w^{-1} \Delta_w$  (recall  $W \subset \Delta_W$ ). The main result is

**THEOREM 4.1.** *If  $w \in W, \omega \in V^*$ , then in  $\text{End } S(V)$ ,*

$$[w^{-1} \Delta_w, \omega^*] = \sum_{w' \xrightarrow{\gamma} w} (w'^{-1}(\gamma)^v, \omega) w^{-1} \Delta_{w'} .$$

We will now fix a reduced decomposition  $w = s_{\alpha_1} \dots s_{\alpha_k}$  and write  $s_i$  for  $s_{\alpha_i}$  and  $w_i = s_{\alpha_n} \dots s_{\alpha_i}$ . First we have the following easy observation.

**LEMMA 4.2.** *Let  $\theta_i = s_k \dots s_{i+1}(\alpha_i) = w_{i+1}(\alpha_i), 1 \leq i \leq k$ . Then*

(i)  $w^{-1} \Delta_w = \Delta_{\theta_1} \Delta_{\theta_2} \dots \Delta_{\theta_k}$

and

(ii)  $s_{\theta_i} (w_i^\wedge)^{-1} = w^{-1}$

*Proof.* Note by (2.2 ii, iv)  $s_\alpha \Delta_\alpha = \Delta_\alpha$ . Hence

$$\begin{aligned} w^{-1} \Delta_w &= s_k \dots s_1 \Delta_{\alpha_1} \dots \Delta_{\alpha_k} = \Delta_{s_k \dots s_1 (\alpha_1)} s_k \dots s_2 \Delta_{\alpha_2} \dots \Delta_{\alpha_k} \\ &= \Delta_{\theta_1} w_2 \Delta_{\alpha_2} \dots \Delta_{\alpha_k} \end{aligned}$$

and induction completes the argument. The second remark follows precisely as in (1.3).

*Proof of (4.1).* We compute

$$\begin{aligned} [w^{-1} \Delta_w, \omega^*] &= [\Delta_{\theta_1} \circ \dots \circ \Delta_{\theta_k}, \omega^*] \\ &= \sum_{j=1}^k \Delta_{\theta_1} \dots \Delta_{\theta_{j-1}} [\Delta_{\theta_j}, \omega^*] \dots \Delta_{\theta_k} . \end{aligned}$$

Let us call the  $j$ -th summand  $P_j$ . Firstly, observe that  $[\Delta_{\theta_j}, \omega^*] = (\theta_j^v, \omega) s_{\theta_j}$  by (2.2 vii). If we substitute this into  $P_j$  and drag the reflection  $s_{\theta_j}$  over to the left we get

$$\begin{aligned} P_j &= \Delta_{\theta_1} \dots \Delta_{\theta_{j-1}} [\Delta_{\theta_j}, \omega^*] \Delta_{\theta_{j+1}} \dots \Delta_{\theta_k} \\ &= (\theta_j^v, \omega) \Delta_{\theta_1} \dots \Delta_{\theta_{j-1}} s_{\theta_j} \Delta_{\theta_{j+1}} \dots \Delta_{\theta_k} \\ &= (\theta_j^j, \omega) s_{\theta_j} \Delta_{s_{\theta_j}(\theta_1)} \dots \Delta_{s_{\theta_j}(\theta_{j-1})} \Delta_{\theta_{j+1}} \dots \Delta_{\theta_k} \\ &= (\theta_j^v, \omega) s_{\theta_j} (w_j^\wedge)^{-1} \Delta_{w_j^\wedge} . \end{aligned}$$

To see this final identity we must argue, by (4.2), that  $s_{\theta_j}(\theta_i) = \pm \theta_{i, \hat{j}}$  where  $\theta_{i, \hat{j}} = s_k \dots \hat{s}_j \dots s_{i+1}(\alpha_i)$ . (As in the above remark,  $\theta_{i, \hat{j}}$  is the  $\theta_i$  for  $w_j^\wedge = s_1 \dots \hat{s}_j \dots s_k$ .) But, we can assume  $i < j$ , in which case

$$\begin{aligned} s_{\theta_j}(\theta_i) &= s_k \dots s_{j+1} s_j s_{j+1} \dots s_k (s_k \dots s_j s_{j+1} \dots s_{i+1}(\alpha_i)) \\ &= s_k \dots \hat{s}_j \dots s_{i+1}(\alpha_i) = \theta_{i, \hat{j}} . \end{aligned}$$

And now, by (4.2 ii)

$$P_j = (\theta_j^v, \omega) w^{-1} \Delta_{w_j^\wedge}$$

and  $s_{w_j^\wedge(\theta_j)}(w_j^\wedge) = w$ , so  $w_j^\wedge \xrightarrow{w_j^\wedge(\theta_j)} w$ . Finally, (1.2) allows us to reindex by the immediate subwords of  $w$

$$\sum_{j=1}^k P_j = \sum_{\substack{\gamma \\ w' \rightarrow w}} ((w')^{-1}(\gamma)^v, \omega) w^{-1} \Delta_{w'}$$

and the proof is complete.

COROLLARY 4.3.

$$\Delta_w \cdot \omega^* = w \cdot \omega^* \cdot w^{-1} \Delta_w + \sum_{\substack{\gamma \\ w' \rightarrow w}} ((w')^{-1}(\gamma), \omega) \Delta_{w'}$$

*Proof.* Multiply (4.1) by  $w$ .

COROLLARY 4.4.

$$\varepsilon \cdot \Delta_w \cdot \omega^* = \sum_{\substack{\gamma \\ w' \rightarrow w}} ((w')^{-1}(\gamma^v), \omega) \varepsilon \cdot \Delta_{w'}$$

*Proof.* The first term in the right-hand side of (4.3) is annihilated by  $\varepsilon$ . It is now easy to dualize the above and obtain

THEOREM 4.5 (Pieri formula). *If*  $w \in W, \alpha \in \Sigma$ , *then in*  $H_W$

$$X_{s_\alpha} \cdot X_w = \sum_{\substack{\gamma \\ w \rightarrow w'}} (w^{-1}(\gamma)^v, \omega_\alpha) X_{w'}$$

*Proof.* Choose  $A$  such that  $\varepsilon \cdot \Delta_{w'}(A) = \delta_{ww'}$ , for example  $\sigma(X_w)$ . Then, by (3.4 ii)

$$\begin{aligned} X_{s_\alpha} \cdot X_w &= c(\omega_\alpha A) \\ &= \sum_{w' \in W} \varepsilon \Delta_{w'}(\omega_\alpha A) X_{w'} \\ &= \sum_{w' \in W} \varepsilon \cdot \Delta_{w'} \omega_\alpha^*(A) X_{w'} \\ &= \sum_{w' \in W} \left( \sum_{\substack{\gamma \\ g \rightarrow w'}} (g^{-1}(\gamma)^v, \omega_\alpha) \varepsilon \circ \Delta_g(A) \right) X_{w'} \\ &= \sum_{w' \in W} \left( \sum_{\substack{\gamma \\ g \rightarrow w'}} (g^{-1}(\gamma)^v, \omega_\alpha) \delta_{gw} \right) X_{w'} \\ &= \sum_{\substack{\gamma \\ w \rightarrow w'}} (w^{-1}(\gamma)^v, \omega_\alpha) X_{w'} \end{aligned}$$

Of course, it is also possible to rewrite this formula in the following equivalent form.

COROLLARY 4.6.

$$X_{s_\alpha} \cdot X_w = \sum_{\substack{\beta \in \Delta^+ \\ l(ws_\beta) = l(w) + 1}} (\beta^v, \omega_\alpha) X_{ws_\beta}$$

*Proof.* It suffices to note  $\sigma_\gamma w = w'$  if and only if  $w \sigma_{w^{-1}(\gamma)} = w'$ .

*Example.* Recall that in  $H_{\Sigma_3}$  we computed

$$X_{s_\alpha s_\beta} = \frac{1}{3}(-X_{s_\alpha}^2 + X_{s_\beta} X_{s_\alpha} + 2X_{s_\beta}^2) .$$

By (4.6), one can compute

$$\begin{aligned} X_{s_\alpha}^2 &= X_{s_\beta s_\alpha} \\ X_{s_\beta} X_{s_\alpha} &= X_{s_\beta s_\alpha} + X_{s_\alpha s_\beta} \\ X_{s_\beta}^2 &= X_{s_\alpha s_\beta} \end{aligned}$$

and this confirms our earlier computation.

## 5. $H_W$ AS A $W$ -MODULE AND PARABOLICS

If  $(W, S)$  is a Coxeter system and  $\theta \subseteq S$  then  $(W_\theta, \theta)$  is also a Coxeter system [6, p. 20] and  $W_\theta$  is called a *parabolic* subgroup of  $W$ . In addition, it is easy to see that a geometric realization  $(\Delta, \Sigma)$  of  $(W, S)$  can be restricted to a geometric realization of  $(W_\theta, \theta)$ . The collection  $\{W_\theta\}_{\theta \subseteq S}$  of parabolic forms a lattice of  $2^{|S|}$  distinct subgroups where, for example,  $W_\theta \cap W_{\theta'} = W_{\theta \cap \theta'}$ . We will eventually be concerned with the set of left cosets of  $W_\theta$  in  $W$ . We define  $W^\theta = \{w \in W : l(ws) = l(w) + 1 \text{ for all } s \in \theta\}$ . The following basic result is well-known [6, p. 37 and p. 45].

**THEOREM 5.1.** *Every element  $w$  of  $W$  can be uniquely expressed as  $w^\theta \cdot w_\theta$  with  $w^\theta \in W^\theta$ ,  $w_\theta \in W_\theta$  and furthermore  $l(w) = l(w^\theta) + l(w_\theta)$ .*

This immediately yields

**COROLLARY 5.2.**  *$W^\theta$  is a complete set of left coset representations for  $W_\theta$  in  $W$  and furthermore provides an element of the coset of minimal length.*

In this section we analyze the subalgebra  $H_W^{W^\theta}$  of  $W_\theta$ -invariants in  $H_W$ . The most straightforward approach is to compute exactly the action of  $W$  on  $H_W$ . This is easily done by exploiting the computation (4.1).

**THEOREM 5.3.** *The structure of  $H_W$  as a  $W$ -module is given by*

$$s_\alpha \cdot X_w = \begin{cases} X_w & \text{if } l(ws_\alpha) = l(w) + 1 \\ X_w - \sum_{\substack{\gamma \\ ws_\alpha \rightarrow w'}} (s_\alpha w^{-1}(\gamma)^v, \alpha) X_{w'} & \text{if } l(ws_\alpha) = l(w) - 1. \end{cases}$$