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$$\frac{d}{|W|} \equiv \frac{\rho^N}{N!} \pmod{I_W}$$

where ρ is the sum of the fundamental weights. Hence one can attempt to compute the operators Δ_w on ρ^N.

It is possible to develop such formulae and we hope to treat them elsewhere. In particular, one might want to conjecture in the general case that $\rho^N \notin I_W$, maybe even for all ρ in the interior of the fundamental chamber.

4. Pieri formula

Recall that the algebra of operators $\mathbb{A}_W$ was generated by both the Δ_α's and the multiplication operators ω^*. Using the basis constructed in (2.9), if one composes such operators, say $\omega^* \circ \Delta_w$ or $\Delta_w \circ \omega^*$, it should be possible to express them linearly in terms of the operators Δ_g, $g \in W$. Of course, our eventual concern is with the algebra $\mathbb{A}_W$ and

$$\varepsilon \circ \omega^* \cdot \Delta_w = 0 .$$

So, if we compute the commutator $[\Delta_w, \omega^*]$ a quick application of ε will yield a formula for $\varepsilon \cdot \Delta_w \circ \omega^*$. Here we are following the strategy of Bernstein-Gelfand-Gelfand [2]. Essentially, this result is our Pieri formula disguised in its dual form.

In order for the techniques of section 1 and induction to be easily applicable, we work with the slightly modified operator $w^{-1}\Delta_w$ (recall $W \subset \mathbb{A}_W$). The main result is

THEOREM 4.1. *If* $w \in W, \omega \in V^*$, *then in* End $S(V)$,

$$[w^{-1}\Delta_w, \omega^*] = \sum_{w' \xrightarrow{\gamma} w} \left(w'^{-1}(\gamma)^v, \omega\right) w^{-1}\Delta_{w'} .$$

We will now fix a reduced decomposition $w = s_{\alpha_1}, ..., s_{\alpha_k}$ and write s_i for s_{α_i} and $w_i = s_{\alpha_n} ... s_{\alpha_i}$. First we have the following easy observation.

LEMMA 4.2. *Let* $\theta_i = s_k ... s_{i+1}(\alpha_i) = w_{i+1}(\alpha_i)$, $1 \leqslant i \leqslant k$. *Then*

(i) $w^{-1}\Delta_w = \Delta_{\theta_1}\Delta_{\theta_2} \cdots \Delta_{\theta_k}$

and

(ii) $s_{\theta_i}(w\hat{_i})^{-1} = w^{-1}$

Proof. Note by (2.2 ii, iv) $s_\alpha \Delta_\alpha = \Delta_\alpha$. Hence

$$w^{-1} \Delta_w = s_k \dots s_1 \Delta_{\alpha_1} \dots \Delta_{\alpha_k} = \Delta_{s_k \dots s_1 (\alpha_1)} s_k \dots s_2 \Delta_{\alpha_2} \dots \Delta_{\alpha_k}$$
$$= \Delta_{\theta_1} w_2 \Delta_{\alpha_2} \dots \Delta_{\alpha_k}$$

and induction completes the argument. The second remark follows precisely as in (1.3).

Proof of (4.1). We compute

$$[w^{-1}\Delta_w, \omega^*] = [\Delta_{\theta_1} \circ \dots \circ \Delta_{\theta_k}, \omega^*]$$
$$= \sum_{j=1}^{k} \Delta_{\theta_1} \dots \Delta_{\theta_{j-1}} [\Delta_{\theta_j}, \omega^*] \dots \Delta_{\theta_k} \ .$$

Let us call the j-th summand P_j. Firstly, observe that $[\Delta_{\theta_j}, \omega^*] = (\theta_j^v, \omega)\, s_{\theta_j}$ by (2.2 vii). If we substitute this into P_j and drag the reflection s_{θ_j} over to the left we get

$$P_j = \Delta_{\theta_1} \dots \Delta_{\theta_{j-1}} [\Delta_{\theta_j}, \omega^*] \Delta_{\theta_{j+1}} \dots \Delta_{\theta_k}$$
$$= (\theta_j^v, \omega) \Delta_{\theta_1} \dots \Delta_{\theta_{j-1}} s_{\theta_j} \Delta_{\theta_{j+1}} \dots \Delta_{\theta_k}$$
$$= (\theta_v^j, \omega)\, s_{\theta_j} \Delta_{s_{\theta_j}(\theta_1)} \dots \Delta_{s_{\theta_j}(\theta_{j-1})} \Delta_{\theta_{j+1}} \dots \Delta_{\theta_k}$$
$$= (\theta_j^v, \omega)\, s_{\theta_j} (w_{\hat{j}})^{-1} \Delta_{w_{\hat{j}}} \ .$$

To see this final identity we must argue, by (4.2), that $s_{\theta_j}(\theta_i) = \pm\, \theta_{i,\hat{j}}$ where $\theta_{i,\hat{j}} = s_k \dots \hat{s}_j \dots s_{i+1} (\alpha_i)$. (As in the above remark, $\theta_{i.\hat{j}}$ is the θ_i for $w_{\hat{j}} = s_1 \dots \hat{s}_j \dots s_k$.) But, we can assume $i < j$, in which case

$$s_{\theta_j}(\theta_i) = s_k \dots s_{j+1} s_j s_{j+1} \dots s_k \big(s_k \dots s_j s_{j+1} \dots s_{i+1} (\alpha_i)\big)$$
$$= s_k \dots \hat{s}_j \dots s_{i+1} (\alpha_i) = \theta_{i,\hat{j}} \ .$$

And now, by (4.2 ii)

$$P_j = (\theta_j^v, \omega)\ w^{-1} \Delta_{w_{\hat{j}}}$$

and $s_{w_{\hat{j}}(\theta_j)} (w_{\hat{j}}) = w$, so $w_{\hat{j}} \xrightarrow{w_{\hat{j}}(\theta_j)} w$. Finally, (1.2) allows us to reindex by the immediate subwords of w

$$\sum_{j=1}^{k} P_j = \sum_{w' \xrightarrow{\gamma} w} \big((w')^{-1} (\gamma)^v, \omega\big)\, w^{-1} \Delta_{w'}$$

and the proof is complete.

COROLLARY 4.3.

$$\Delta_w \cdot \omega^* = w \cdot \omega^* \cdot w^{-1} \Delta_w + \sum_{w' \xrightarrow{\gamma} w} ((w')^{-1}(\gamma)|, \omega(\Delta_{w'} .$$

Proof. Multiply (4.1) by w.

COROLLARY 4.4.

$$\varepsilon \cdot \Delta_w \cdot \omega^* = \sum_{w' \xrightarrow{\gamma} w} ((w')^{-1}(\gamma^v), \omega)\, \varepsilon \cdot \Delta_{w'} .$$

Proof. The first term in the right-hand side of (4.3) is annihilated by ε. It is now easy to dualize the above and obtain

THEOREM 4.5 (Pieri formula). *If* $w \in W, \alpha \in \Sigma,$ *then in* H_W

$$X_{s_\alpha} \cdot X_w = \sum_{w \xrightarrow{\gamma} w'} (w^{-1}(\gamma)^v, w_\alpha) X_{w'} .$$

Proof. Choose A such that $\varepsilon \cdot \Delta_{w'}(A) = \delta_{ww'}$, for example $\sigma(X_w)$. Then, by (3.4 ii)

$$\begin{aligned}
X_{s_\alpha} \cdot X_w &= c(\omega_\alpha A) \\
&= \sum_{w' \in W} \varepsilon \Delta_{w'}(\omega_\alpha A) X_{w'} \\
&= \sum_{w' \in W} \varepsilon \cdot \Delta_{w'} \omega_\alpha^*(A) X_{w'} \\
&= \sum_{w' \in W} \Big(\sum_{g \xrightarrow{\gamma} w'} (g^{-1}(\gamma)^v, \omega_\alpha)\, \varepsilon \circ \Delta_g(A) \Big) X_{w'} \\
&= \sum_{w' \in W} \Big(\sum_{g \xrightarrow{\gamma} w'} (g^{-1}(\gamma)^v, \omega_\alpha)\, \delta_{gw} \Big) X_{w'} \\
&= \sum_{w \xrightarrow{\gamma} w'} (w^{-1}(\gamma)^v, \omega_\alpha) X_{w'} .
\end{aligned}$$

Of course, it is also possible to rewrite this formula in the following equivalent form.

COROLLARY 4.6.

$$X_{s_\alpha} \cdot X_w = \sum_{\substack{\beta \in \Delta + \\ l(ws_\beta) = l(w) + 1}} (\beta^v, \omega_\alpha) X_{ws_\beta} .$$

Proof. It suffices to note $\sigma_\gamma w = w'$ if and only if $w \sigma_{w^{-1}(\gamma)} = w'$.

Example. Recall that in H_{Σ_3} we computed

$$X_{s_\alpha s_\beta} = \tfrac{1}{3}(-X^2_{s_\alpha} + X_{s_\beta} X_{s_\alpha} + 2X^2_{s_\beta}) \ .$$

By (4.6), one can compute

$$X^2_{s_\alpha} = X_{s_\beta s_\alpha}$$

$$X_{s_\beta} X_{s_\alpha} = X_{s_\beta s_\alpha} + X_{s_\alpha s_\beta}$$

$$X^2_{s_\beta} = X_{s_\alpha s_\beta}$$

and this confirms our earlier computation.

5. H_W AS A W-MODULE AND PARABOLICS

If (W, S) is a Coxeter system and $\theta \subseteq S$ then (W_θ, θ) is also a Coxeter system [6, p. 20] and W_θ is called a *parabolic* subgroup of W. In addition, it is easy to see that a geometric realization (Δ, Σ) of (W, S) can be restricted to a geometric realization of (W_θ, θ). The collection $\{W_\theta\}_{\theta \subset S}$ of parabolic forms a lattice of $2^{|S|}$ distinct subgroups where, for example, $W_\theta \cap W_{\theta'} = W_{\theta \cap \theta'}$. We will eventually be concerned with the set of left cosets of W_θ in W. We define $W^\theta = \{w \in W : l(ws) = l(w) + 1 \text{ for all } s \in \theta\}$. The following basic result is well-known [6, p. 37 and p. 45].

THEOREM 5.1. *Every element w of W can be uniquely expressed as $w^\theta \cdot w_\theta$ with $w^\theta \in W^\theta$, $w_\theta \in W_\theta$ and furthermore $l(w) = l(w^\theta) + l(w_\theta)$.*

This immediately yields

COROLLARY 5.2. *W^θ is a complete set of left coset representations for W_θ in W and furthermore provides an element of the coset of minimal length.*

In this section we analyze the subalgebra $H_W^{W_\theta}$ of W_θ-invariants in H_W. The most straightforward approach is to compute exactly the action of W on H_W. This is easily done by exploiting the computation (4.1).

THEOREM 5.3. *The structure of H_W as a W-module is given by*

$$s_\alpha \cdot X_w = \begin{cases} X_w & \text{if } l(ws_\alpha) = l(w) + 1 \\ X_w - \displaystyle\sum_{ws_\alpha \xrightarrow{\gamma} w'} (s_\alpha w^{-1}(\gamma)^\vee, \alpha) X_{w'} & \text{if } l(ws_\alpha) = l(w) - 1. \end{cases}$$