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*Example.* Recall that in  $H_{\Sigma_3}$  we computed

$$X_{s_\alpha s_\beta} = \frac{1}{3}(-X_{s_\alpha}^2 + X_{s_\beta} X_{s_\alpha} + 2X_{s_\beta}^2) .$$

By (4.6), one can compute

$$\begin{aligned} X_{s_\alpha}^2 &= X_{s_\beta s_\alpha} \\ X_{s_\beta} X_{s_\alpha} &= X_{s_\beta s_\alpha} + X_{s_\alpha s_\beta} \\ X_{s_\beta}^2 &= X_{s_\alpha s_\beta} \end{aligned}$$

and this confirms our earlier computation.

## 5. $H_W$ AS A $W$ -MODULE AND PARABOLICS

If  $(W, S)$  is a Coxeter system and  $\theta \subseteq S$  then  $(W_\theta, \theta)$  is also a Coxeter system [6, p. 20] and  $W_\theta$  is called a *parabolic* subgroup of  $W$ . In addition, it is easy to see that a geometric realization  $(\Delta, \Sigma)$  of  $(W, S)$  can be restricted to a geometric realization of  $(W_\theta, \theta)$ . The collection  $\{W_\theta\}_{\theta \subseteq S}$  of parabolic forms a lattice of  $2^{|S|}$  distinct subgroups where, for example,  $W_\theta \cap W_{\theta'} = W_{\theta \cap \theta'}$ . We will eventually be concerned with the set of left cosets of  $W_\theta$  in  $W$ . We define  $W^\theta = \{w \in W : l(ws) = l(w) + 1 \text{ for all } s \in \theta\}$ . The following basic result is well-known [6, p. 37 and p. 45].

**THEOREM 5.1.** *Every element  $w$  of  $W$  can be uniquely expressed as  $w^\theta \cdot w_\theta$  with  $w^\theta \in W^\theta$ ,  $w_\theta \in W_\theta$  and furthermore  $l(w) = l(w^\theta) + l(w_\theta)$ .*

This immediately yields

**COROLLARY 5.2.**  *$W^\theta$  is a complete set of left coset representations for  $W_\theta$  in  $W$  and furthermore provides an element of the coset of minimal length.*

In this section we analyze the subalgebra  $H_W^{W^\theta}$  of  $W_\theta$ -invariants in  $H_W$ . The most straightforward approach is to compute exactly the action of  $W$  on  $H_W$ . This is easily done by exploiting the computation (4.1).

**THEOREM 5.3.** *The structure of  $H_W$  as a  $W$ -module is given by*

$$s_\alpha \cdot X_w = \begin{cases} X_w & \text{if } l(ws_\alpha) = l(w) + 1 \\ X_w - \sum_{\substack{\gamma \\ ws_\alpha \rightarrow w'}} (s_\alpha w^{-1}(\gamma)^v, \alpha) X_{w'} & \text{if } l(ws_\alpha) = l(w) - 1. \end{cases}$$

*Proof.* As in (4.5), choose  $A$  such that  $\varepsilon \Delta_g(A) = \delta_{gw}$ . Then, since  $c$  is a  $W$ -map

$$\begin{aligned}
 s_\alpha X_w &= c(s_\alpha A) = \sum_{w' \in W} \varepsilon \Delta_{w'}(s_\alpha A) X_{w'} \\
 &= \sum_{w' \in W} \varepsilon \Delta_{w'}(1 - \alpha^* \Delta_\alpha)(A) X_{w'} \\
 &= X_w - \sum_{w' \neq w} (\varepsilon \Delta_{w'} \alpha^*) \Delta_\alpha(A) X_{w'} \\
 &= X_w - \sum_{\substack{\gamma \\ g \xrightarrow{\gamma} w'}} (g^{-1}(\gamma)^v, \alpha) \varepsilon \Delta_{gs_\alpha}(A) X_{w'} && \text{by (4.1)} \\
 &= \sum_{\substack{\gamma \\ g \xrightarrow{\gamma} w' \\ l(gs_\alpha) = l(g) + 1 \\ gs_\alpha = w}} (g^{-1}(\gamma)^v, \alpha) X_{w'} \\
 &= X_w - \sum_{\substack{\gamma \\ ws_\alpha \xrightarrow{\gamma} w'}} (s_\alpha w^{-1}(\gamma)^v, \alpha) X_{w'}
 \end{aligned}$$

Note, that the summation in the next to the last line is non-vacuous if and only if  $l(ws_\alpha) = l(w) - 1$ . This completes the proof.

**COROLLARY 5.4.**  $X_w \in H_W^{W^\theta}$  if  $w \in W^\theta$ .

*Proof.* Immediate from (5.3) and the definition of  $W^\theta$ .

The following elementary result shows that the  $X_w, w \in W^\theta$ , are actually an  $\mathbf{R}$ -basis for  $H_W^{W^\theta}$ .

**LEMMA.** *If a finite group  $G$  acts on a real vector space  $V$  via the regular representation and  $H$  is a subgroup of  $G$ , then*

$$\dim_{\mathbf{R}}(V^H) = [G : H].$$

*Proof.* Let  $\{e_g\}_{g \in G}$  be a basis for  $V$ , so that

$$g' \cdot e_g = e_{gg'}$$

Then if  $\chi = \sum_{g \in G} \xi_g \in V^H$ , we claim  $\xi_g = \xi_{g'}$ , if  $g \equiv g' \pmod{H}$ . Indeed, if  $g = g' h, h \in H$ , then

$$\begin{aligned}
 \xi_{g'} &= \text{coefficient of } e_{g'} \chi \text{ in} \\
 &= \text{coefficient of } e_{g'} \text{ in } h^{-1} \chi \\
 &= \text{coefficient of } e_{g'h} \text{ in } \chi \\
 &= \xi_g.
 \end{aligned}$$

Hence, there are at most  $[G : H]$  free parameters in determining  $\chi \in V^H$  and clearly each choice gives an invariant. This finishes the proof.

**COROLLARY 5.6.**  $\dim(H_W^{W^\theta}) = [W : W_\theta] = |W^\theta|$  and the  $X_w$ ,  $w \in W^\theta$ , are an  $\mathbf{R}$ -basis for  $H_W^{W^\theta}$ .

*Proof.* Chevalley [8] has shown that  $S_W$ , hence  $H_W$ , is abstractly equivalent to the regular representation of  $W$ , as a  $W$ -module. Hence, (5.5) applies and the result follows.

It is now possible to "restrict" the Pieri formula (4.5) for  $H_W$  to  $H_W^{W^\theta}$ . We have

**THEOREM 5.7.** If  $w, w' \in W^\theta$  and in  $H_W$

$$\begin{aligned} X_w \cdot X_{w'} &= \sum_{w'' \in W} c(w, w', w'') X_{w''} \\ \text{then in } H_W^{W^\theta} \quad X_w \cdot X_{w'} &= \sum_{w'' \in W^\theta} c(w, w', w'') X_{w''} \end{aligned}$$

*Proof.* One need only observe that the vector space map  $r : H_W \rightarrow H_W^{W^\theta}$  given by

$$r(X_w) = \begin{cases} X_w & \text{if } w \in W^\theta \\ 0 & \text{otherwise} \end{cases}$$

is a retraction. Then, applying  $r$  to both sides of the first equation yields the second equation since the invariants form a subalgebra.

This result will be useful in the next section for computing inside the algebra of  $W_\theta$ -invariants. Notice that an appropriate Giambelli formula for  $H_W^{W^\theta}$  is not as easily obtained. This is because the Giambelli formula for  $H_W$  gives  $X_w$  as a polynomial in the  $X_{s_\alpha}$ 's and not all of these are in the algebra  $H_W^{W^\theta}$ , so this is not quite the right thing.

## 6. APPLICATION:

### THE COMBINATORICS OF THE CLASSICAL PIERI FORMULA

In the last section we saw that given a pair  $(W, W_\theta)$  of a Coxeter group and a parabolic subgroup, one could construct a formula to describe the multiplication of Schubert generators in the invariant subalgebra  $H_W^{W^\theta}$ . In this section, we examine the particular case  $(\Sigma_{n+k}, \Sigma_k \times \Sigma_n)$  where  $\Sigma_m$  denotes the symmetric group on  $m$  letters. Indeed,  $\Sigma_{n+k}$  is the Weyl group of the root system of type  $A_{n+k-1}$ , which we recall briefly here. Let  $V' = \mathbf{R}^{n+k}$