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Hence, there are at most $[G : H]$ free parameters in determining $\chi \in V^H$ and clearly each choice gives an invariant. This finishes the proof.

COROLLARY 5.6. *dim* $(H_W^{W_\theta}) = [W : W_\theta] = |W^\theta|$ *and the* X_w, $w \in W^\theta$, *are an* **R**-*basis for* $H_W^{W_\theta}$.

Proof. Chevalley [8] has shown that S_W, hence H_W, is abstractly equivalent to the regular representation of W, as a W-module. Hence, (5.5) applies and the result follows.

It is now possible to "restrict" the Pieri formula (4.5) for H_W to $H_W^{W_\theta}$. We have

THEOREM 5.7. *If* $w, w' \in W^\theta$ *and in* H_W

$$X_w \cdot X_{w'} = \sum_{w'' \in W} c(w, w', w'') X_{w''}$$

then in $H_W^{W_\theta}$

$$X_w \cdot X_{w'} = \sum_{w'' \in W^\theta} c(w, w', w'') X_{w''}$$

Proof. One need only observe that the vector space map $r : H_W \to H_W^{W_\theta}$ given by

$$r(X_w) = \begin{cases} X_w & \text{if } w \in W^\theta \\ 0 & \text{otherwise} \end{cases}$$

is a retraction. Then, applying r to both sides of the first equation yields the second equation since the invariants form a subalgebra.

This result will be useful in the next section for computing inside the algebra of W_θ-invariants. Notice that an appropriate Giambelli formula for $H_W^{W_\theta}$ is not as easily obtained. This is because the Giambelli formula for H_W gives X_w as a polynomial in the X_{s_α}'s and not all of these are in the algebra $H_W^{W_\theta}$, so this is not quite the right thing.

6. APPLICATION: THE COMBINATORICS OF THE CLASSICAL PIERI FORMULA

In the last section we saw that given a pair (W, W_θ) of a Coxeter group and a parabolic subgroup, one could construct a formula to describe the multiplication of Schubert generators in the invariant subalgebra $H_W^{W_\theta}$. In this section, we examine the particular case $(\Sigma_{n+k}, \Sigma_k \times \Sigma_n)$ where Σ_m denotes the symmetric group on m letters. Indeed, Σ_{n+k} is the Weyl group of the root system of type A_{n+k-1}, which we recall briefly here. Let $V' = \mathbf{R}^{n+k}$

equipped with the usual inner product and let $e_1, ..., e_{n+k}$ denote the standard basis. Σ_{n+k} acts by permuting these basic elements. This action is effective on the $(n+k-1)$-dimensional subspace

$$V = \left\{ \sum_{i=1}^{n+k} \lambda_i e_i : \sum_{i=1}^{n+k} \lambda_i = 0 \right\}$$

and it is easy to see $\Delta = \{e_i - e_j\}_{j \neq i}$ can be chosen as the corresponding root system. In addition, the simple roots $\Sigma = \{e_i - e_{i+1}\}_{1 \leq i \leq n+k-1}$ and the positive roots $\Delta^+ = \{e_i - e_j\}_{i<j}$, induce the usual transpositions of the basis vectors.

The main result of this section is the identification of the Pieri formula for $H_{\Sigma_{n+k}}^{\Sigma_k \times \Sigma_n}$ with the classical Pieri formula (see [7, 16]).

We begin with a rapid review of Chern's Schubert calculus for the cohomology of a complex grassmannian [7]. Let $G_k(\mathbf{C}^{n+k})$ denote the space of k-dimensional complex subspaces in $\mathbf{C}^{n+k}$. This is a compact, smooth manifold of dimension $2nk$. Ehresmann [14] described a cell-decomposition for $G_k(\mathbf{C}^{n+k})$ (along with other algebraic homogeneous spaces) whose cells are identified by certain Schubert symbols $(d_1, ..., d_k)$, where

$$1 \leqslant d_1 < ... < d_k \leqslant n+k .$$

Each symbol yields a cohomology class $\langle d_1 ... d_k \rangle$ of dimension

$$2 \sum_{i=1}^{k} (d_i - 1) = \left(\sum_{i=1}^{k} d_i \right) - \frac{k(k+1)}{2}$$

Geometrically, $\langle d_1, ..., d_k \rangle$ is the cocycle dual to the cell

$$[d_1, ..., d_k] = \{X \in G_k(\mathbf{C}^{n+k}) : \dim (X \cap \mathbf{R}^{d_i}) \geqslant i\} .$$

It is easy to see the d_i's describe the "jump-points" in the sequence

$$0 \leqslant \dim (X \cap \mathbf{C}^1) \leqslant \dim (X \cap \mathbf{C}^2) \leqslant ... \leqslant \dim (X \cap \mathbf{C}^{n+k-1}) \leqslant n+k$$

where $0 \subseteq \mathbf{C}^1 \subseteq \mathbf{C}^2 \subseteq ... \subseteq \mathbf{C}^{n+k}$ is the standard flag determined by the coordinate axes.

On the other hand, $G_k(\mathbf{C}^{n+k})$ can also be profitably thought of as the homogeneous space G/P where G is the complex Lie group $GL_{n+k}(\mathbf{C})$ and P is the maximal parabolic subgroup of the form

$$\left(\begin{array}{c|c} GL_k(\mathbf{C}) & * \\ \hline 0 & GL_n(\mathbf{C}) \end{array} \right)$$

If K denotes the maximal compact subgroup U_{n+k} of G then we also have the identification $G_k(\mathbf{C}^{n+k}) = K/(U_k \times U_n)$.

More generally, one can consider a complex semisimple Lie group G and a parabolic P_θ in G corresponding to a subset θ of the simple roots Σ. The homogeneous space G/P_θ has been studied by various authors and we will assume known that

$$H^*(G/P_\theta; \mathbf{R}) \cong H^*(G/B; \mathbf{R})^{W_\theta} \cong (S_W)^{W_\theta} .$$

This will be the basic topological input [2].

Now we fix G to be the Lie group of type A_{n+k-1} and $\theta = \Sigma - \{\alpha_k\}$ (where we write $\alpha_j = e_j - e_{j+1}$ and $s_j = s_{\alpha_j}$) so that $G_k(\mathbf{C}^{n+k}) \cong G/P_\theta$. We begin with some easy length computations.

LEMMA 6.1. *If* $w \in W$, *then*

$$l(ws_{ij}) - l(w) = p_{ij}(2\,|I_{i,j}|+1)$$

where

$$p_{ij} = \begin{cases} +1 & w(i) < w(j) \\ -1 & w(i) > w(j) \end{cases}$$

and

$$I_{i,j} = \{i < z < j : w(z) \text{ is between } w(i) \text{ and } w(j)\} .$$

In particular, $l(ws_{ij}) = l(w) + 1$ *if and only if* (i) $w(i) < w(j)$ *and* (ii) *there are no intermediate* w*-values, i.e.* $I_{i,j} = \phi$ *(we often abbreviate this pair of conditions by* $w(i) << w(j)$*).*

Proof. Recall the length function on $\sum_{n+k}$ is given by $l(w) = \sum_{j=1}^{n-1} e_j(w)$, where $e_j = |\{i > j : w(i) < w(j)\}|$, the number of inversions of w. Hence

$$l(ws_{ij}) - l(w) = (e_i' - e_i) + (e_j' - e_j) + \sum_{i<z<j} (e_z' - e_z)$$

where $e_l = e_l(w)$ and $e_l = e_l(ws_{ij})$. Certainly, right multiplication by s_{ij} does not affect the values of e_l below i or above j. Also

$$e_i' = e_j + |\{i \leqslant z < j : w(z) < w(j)\}| = e_j + e$$
$$e_j' = e_i - |\{i < z \leqslant j : w(z) < w(i)\}| = e_j - \bar{e}$$

so we get

$$(e_i' - e_i) + (e_j' - e_j) = (e_j + e - e_i) + (e_i - \bar{e} - e_j) = e - \bar{e} = p_{i,j}(|I_{i,j}|+1)$$

It is easy to see

$$e_z' - e_z = \begin{cases} p_{i,j} & \text{if } z \in I_{i,j} \\ 0 & \text{otherwise} \end{cases}$$

putting this all together we get the result. The second assertion follows immediately.

We can now write down (4.6) for H_W, $W = \sum_{n+k}$

PROPOSITION 6.2. *If* $w \in \sum_{n+k}$, $1 \leqslant i \leqslant n+k-1$, *then in* $H_{\Sigma_{n+k}}$

$$X_{s_i} \cdot X_w = \sum_{(b,t)} X_{ws_{bt}}$$

where (b, t) *satisfies* $b \leqslant i < t$ *and* $w(b) << w(t)$.

Proof. By (4.6), $X_{ws_{bt}}$ appears with coefficient $((e_b - e_t)^v, \omega_i)$ if and only if $l(ws_{bt}) = l(w) + 1$. This is equivalent to the last condition by (6.1). Finally $(e_b - e_t)^v = e_b - e_t = \alpha_b + \ldots + \alpha_{t-1}$, so that first condition is also needed and the coefficient is correct.

Remark. The Poincaré dual of this formula appears in [18, p. 265].

We now identify the set of left coset representatives W^θ. If $1 \leqslant d_1 < \ldots < d_k \leqslant n+k$ and $d_1' < \ldots < d_n'$ is an ordered enumeration of the complement then we define $(d_1, \ldots, d_k) \in \sum_{n+k}$, by

$$(d_1, \ldots, d_k)(i) = \begin{cases} d_i\ 1 \leqslant i \leqslant k \\ d_{i-k}'\ k+1 \leqslant i \leqslant k+n \end{cases} .$$

(We also write $X(d_1, \ldots, d_k)$ when it is convenient.)

LEMMA 6.3.

$$W^\theta = \{(d_1, \ldots, d_k) : 1 \leqslant d_1 < \ldots < d_k \leqslant n+k\}$$

and $l(d_1, \ldots, d_k) = \sum_{j=1} (d_j - j)$.

Proof. Clearly $l((d_1, \ldots, d_k)s_i) = l(d_1, \ldots, d_k) + 1$, for all $i \neq k$ by (6.1), for example. Since $|W^\theta| = |W| / |W_\theta| = \binom{n+k}{k}$ the first assertion follows. For the second, we need only observe

$$e_j(d_1, \ldots, d_k) = \begin{cases} d_j - j & \text{if } j \leqslant k \\ 0 & \text{otherwise.} \end{cases}$$

This lemma indicates how the Schubert notation arises from a group-theoretic point of view. That this notation is consistent with the geometry is a theorem of Demazure [12].

A Pieri formula should compute the product of X_w (a linear generator, by (5.6)) and an algebraic generator. Since the map

$$S(V)^{W_\theta} \xrightarrow{c} (H_W)^{W_\theta}$$

is onto we can find algebraic generators by computing the images of W_θ-invariants. In general, W_θ is a (reducible) Coxeter group, so we have the fundamental invariants [20]. In our case, we have simply

$$S(V)^{W_\theta} = \mathbf{Z}[\tau_1, \ldots, \tau_k, \sigma_1, \ldots, \sigma_n]$$

where $\tau_i = s_i(e_1, \ldots, e_k)$, $1 \leqslant i \leqslant k$, and $\sigma_j = s_j(e_{k+1}, \ldots, e_{k+n})$, $1 \leqslant j \leqslant n$ and s_j denotes the j^{th} elementary symmetric function in an appropriate number of variables. One knows $c(\sigma_j)$ suffices to generate $H_W^{W_\theta}$. So we compute

LEMMA 6.4.

$$c(\sigma_j) = (-1)^j X_{s_{k+j-1}, \ldots, s_k} = (-1)^j X(1, 2, \ldots, k-1, k+j) \quad .$$

Proof. By section 2

$$c(\sigma_j) = \sum_{l(w) = j} \Delta_w(\omega_j) X_w$$

If we write Δ_t for Δ_{α_t}, then clearly $\Delta_t(\sigma_j) = 0$, if $t \neq k$ and

$$\begin{aligned} \Delta_k(\sigma_j) &= \frac{s_j(e_{k+1}, \ldots, e_{k+n}) - s_j(e_k, \ldots, e_{k+n})}{e_k - e_{k+1}} \\ &= \frac{(e_{k+1} - e_k) s_{j-1}(e_{k+2}, \ldots, e_{k+n})}{e_k - e_{k+1}} \\ &= (-1) s_{j-1}(e_{k+2}, \ldots, e_{k+n}) \end{aligned}$$

We can continue by induction and get $\Delta_{k+j-1} \ldots \Delta_k(\sigma_j) = (-1)^j$, while any other sequence of simple roots yields zero.

We now proceed to a computation of

$$X(1, 2, \ldots, k-1, k+j) X(d_1, \ldots, d_k) .$$

The case $j = 1$ is easy.

PROPOSITION 6.5.

$$X(1, 2, \ldots, k-1, k+1) X(d_1, \ldots, d_k) = \sum_{d_i + 1 < d_{i+1}} X(d_1, \ldots, d_i + 1, \ldots, d_k) \ .$$

Proof. Since $(1, 2, \ldots, k-1, k+1) = s_k$, we apply the case $i = k$ of (6.2) and observe $w(b) \ll w(t)$ if and only if $w(t) = w(b) + 1$.

We now observe

LEMMA 6.6.

$$c(\sigma_j) = s_j(X_{s_{k+1}} - X_{s_k}, X_{s_{k+2}} - X_{s_{k+1}}, \ldots, -X_{s_{n+k-1}})$$

Proof. By the tables of [6], the i-th fundamental weight is

$$\omega_i = e_1 + \ldots + e_i - \left(\frac{i}{n+k}\right) \sigma_1 (e_1, \ldots, e_{n+k}) \ .$$

Hence $\omega_i \equiv e_1 + \ldots + e_i \pmod{I_w}$ and we get

$$\begin{aligned} c(\sigma_j) &= c\left(s_j(e_{k+1}, \ldots, e_{k+n})\right) \\ &= c\left(s_j(\omega_{k+1} - \omega_k, \ldots, -\omega_{n+k-1})\right) \\ &= s_j(X_{s_{k+1}} - X_{s_k}, \ldots, -X_{s_{n+k-1}}) \end{aligned}$$

since c kills I_W and (3.4 ii).

This suggests the following computation.

LEMMA 6.7. *For all* $i, k+1 \leqslant i \leqslant k+n, w \in W$; *in* H_W

$$(X_{s_i} - X_{s_{i+1}}) X_w = \sum_{\substack{i < t \\ w(i) << w(t)}} X_{ws_{it}} - \sum_{\substack{k < b < i \\ w(b) << w(i)}} X_{ws_{bi}} - \sum_{\substack{b \leqq k \\ w(b) << w(i)}} X_{ws_{bi}}$$

Proof. Computing with (6.2), we get

$$X_{s_i} X_w = \sum_{\substack{b \leqq i-1 \\ i < t \\ w(b) << w(t)}} X_{ws_{bt}} + \sum_{\substack{i < t \\ w(i) << w(t)}} X_{ws_{it}}$$

and

$$X_{s_{i-1}} X_w = \sum_{\substack{b \leqq i-1 \\ i<t \\ w(b) << w(t)}} X_{ws_{bt}} + \sum_{\substack{b<i \\ w(b) << w(i)}} X_{ws_{bi}}$$

Upon subtracting and breaking up the second term the desired expression follows.

THEOREM 6.8. *In*

$$s_j (X_{s_{k+1}} - X_{s_k}, \ldots, -X_{s_{n+k-1}}) X(d_1, \ldots, d_k) = (-1)^j \Sigma X(e_1, \ldots, e_k)$$

where the summation ranges over $(e_1, \ldots, e_k)$ *satisfying* $d_i \leqslant e_i \leqslant d_{i+1}$ *and* $\sum_{i=1}^{k} e_i = j + \sum_{i=1}^{k} d_i$.

Proof. Of course

$$s_j = \sum_{k+1 \leqq t_1 < \ldots < t_j \leqq k+n} (X_{s_{t_j}} - X_{s_{t_j - 1}}) \ldots (X_{s_{t_1}} - X_{s_{t_1 - 1}})$$

where we set $X_{s_{k+n}} = 0$. It is not difficult to check that the third term of (6.7) alone yields the right-hand side. Hence it remains to show that the contributions arising whenever either of the first two terms of (2.7) are involved cancel in the final summation. To do this it suffices to show that the resulting subscripts in W do not lie in W^θ. (Then they must have coefficient zero since $H_{W^\theta}^{W}$ is a subalgebra of H_w.)

Now the first two terms of (6.7) always give a transposition above $k+1$ and it must be elementary one by (6.1), say s_i, $i \geqslant k$. Such a transposition will send an element of W^θ out of W^θ. We claim no further transposition s_{bt}, with either $b \geqslant i$ or $t \geqslant i$, will put the subscript back in W^θ. Both cases are easy to check and the proof is complete. Finally, by a substitution from (6.4) we get

COROLLARY 6.9. (Pieri formula). *In* $H_{\Sigma_{n+k}}^{\Sigma_k \times \Sigma_n} = H^*(G_k(\mathbf{C}^{n+k}))$

$$X(1, 2, \ldots, k-1, k+j) X(d_1, \ldots, d_k) = \Sigma X(e_1, \ldots, e_k)$$

where the summation is as in (6.8).

Remarks 1. The above formal arguments are equally valid for the Chow ring of the grassmann variety over an arbitrary algebraically closed field.

2. One can hope to mimick the above strategy for the homogeneous space SO_{2n+1}/U_n. The group G is of type B_n and the maximal parabolic is determined by the "right-end" root. It is not difficult to write out the Pieri formula for the flag manifold of type B_n (see the author's "Pieri formulae for classical groups", preprint). In addition, W^θ, for this case, can be identified with the power set of $\{1, 2, ..., n\}$ and one can compute $c(\sigma_j) = 2X_{\{j\}}$. (The 2 occurs because c is not onto.) Still the problem is complicated by multiplicities. We hope to return to this elsewhere.

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