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simple pole and no other singularities. It is easy to see that $f : X \rightarrow \mathbf{P}^1$ is then an isomorphism.

(3.9) COROLLARY. *If $D \sim D'$, then $\deg D = \deg D'$.*

Proof: $D \sim D'$ implies $\mathcal{O}(D) \approx \mathcal{O}(D')$, hence $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D'))$. Hence $\deg D = \deg D'$ by (3.4).

(3.10) *Definition.* The *degree* of a line bundle \mathcal{L} is the degree of any $D \in \text{Div } X$ such that $\mathcal{L} \approx \mathcal{O}(D)$, i.e. the degree of the divisor of any meromorphic section of \mathcal{L} .

(3.11) *Remark.* The above definition is justified by (2.11) and (3.9). It is clear that the map $\deg : \text{Pic } X \rightarrow \mathbf{Z}$ is a group homomorphism.

(3.13) *Definition.* The *degree* of a vector bundle \mathcal{V} is that of the line bundle $\det \mathcal{V} = \bigwedge_{\mathcal{O}_x}^r \mathcal{V}$, $r = \text{rank } \mathcal{V}$.

(3.14) *Remark.* The stalk of $(\det \mathcal{V})^{-1} = \text{Hom}(\det \mathcal{V}, \mathcal{O}_X)$ at any $P \in X$ consists \mathcal{O}_P -multilinear alternate maps $\mathcal{V}_P \times \dots \times \mathcal{V}_P$ (r times) $\rightarrow \mathcal{O}_P$.

(3.15) PROPOSITION. *If $0 \rightarrow \mathcal{V}' \rightarrow \mathcal{V} \rightarrow \mathcal{V}'' \rightarrow 0$ is an exact sequence of vector bundles, then $\deg \mathcal{V} = \deg \mathcal{V}' + \deg \mathcal{V}''$.*

Proof: $\det \mathcal{V} \approx \det \mathcal{V}' \otimes \det \mathcal{V}''$.

(3.16) PROPOSITION. (Riemann-Roch theorem, preliminary form). *For any vector bundle \mathcal{V} on X ,*

$$\chi(\mathcal{V}) = \deg \mathcal{V} + \text{rank } \mathcal{V} \cdot \chi(\mathcal{O})$$

Proof: In view of (3.15), (3.2) and (2.11), the proposition follows from (3.4) by induction on rank \mathcal{V} .

§ 4. THE DEGREE OF THE CANONICAL LINE BUNDLE

Recall that the canonical line bundle K on X is the sheaf of holomorphic 1-forms.

(4.1) THEOREM. $\deg K = 2g - 2 = -2 \chi(\mathcal{O})$.

Proof: Choose any nonconstant meromorphic function for X , and consider the holomorphic map $f: X \rightarrow \mathbf{P}^1 = Y$. Then the *ramification divisor* $R = \sum e(P)P$ of f is defined as follows: for suitable uniformising parameters z and w at P and $f(P)$ respectively, $w(f(z)) = z^{e(P)+1}$. After composing f with a fractional linear transformation if necessary, we may assume that f is unramified over ∞ , i.e. $e(P) = 0$ if $f(P) = \infty$. Note that $r = \sum_{P \in f^{-1}(Q)} (e(P)+1)$ is independent of $Q \in Y$, being clearly the rank of the vector bundle $f_0(\mathcal{O}_X)$ on Y (cf. (2.4)). Now df is a meromorphic 1-form on X (i.e. a meromorphic section of K_X), with zeros of orders $e(P)$ at the P with $f(P) \neq \infty$, and poles of order two at each of the r poles of f . Thus we have:

$$(4.2) \quad (\text{Riemann-Hurwitz formula}). \quad \deg K = \deg R - 2r.$$

On the other hand, by (2.16) and (3.16), we have

$$(4.3) \quad \begin{aligned} \chi(\mathcal{O}_X) &= \chi(f_0(\mathcal{O}_X)) = \deg f_0(\mathcal{O}_X) + r \chi(\mathcal{O}_Y) \\ &= \deg f_0(\mathcal{O}_X) + r. \end{aligned}$$

Thus, to finish the proof of (4.1), we must prove:

$$(4.4) \quad \deg f_0(\mathcal{O}_X) = -\frac{1}{2} \deg R.$$

To prove (4.4), let $\mathcal{L} = \det f_0(\mathcal{O}_X)$. Then we shall show that there is a canonical \mathcal{O}_Y -linear map $\delta: \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{O}_Y$ which, at any $Q \in Y$, looks like multiplication by $t_Q^{\delta(Q)}$, where $\delta(Q) = \sum_{P \in f^{-1}(Q)} e(P)$ (t_Q a uniformising parameter at Q). Since $\sum_Q \delta(Q) = \deg R$, this will prove (4.4).

The map δ is the classical discriminant map. To define it, we first define the "trace" map $\tau: f_0(\mathcal{O}_X) \rightarrow \mathcal{O}_Y$: for $U \subset Y$ open and $h \in \mathcal{O}_X(f^{-1}(U))$, $\tau(h)(Q) = \sum_{P \in f^{-1}(Q)} (e(P)+1)h(P)$ for all $Q \in U$. Then clearly $\tau(h) \in \mathcal{O}_Y(U)$, and τ is \mathcal{O}_Y -linear. Now for any $U \subset Y$ open and any two r -tuples $\lambda = (\lambda_1, \dots, \lambda_r)$, $\mu = (\mu_1, \dots, \mu_r)$ of elements of $\mathcal{O}_X(f^{-1}(U))$ (recall that $r = \text{rank } f_0(\mathcal{O}_X)$), we set $\delta(\lambda, \mu) = \det(\tau(\lambda_i \mu_j))$. Clearly δ is \mathcal{O}_Y -multilinear and alternating in each of λ and μ , hence defines an \mathcal{O}_X -linear map

$$\delta: \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{O}_Y$$

This is the desired map. To compute the effect of δ at any $Q \in Y$, let us assume first that $f^{-1}(Q)$ is a single point P . In suitable coordinate

systems at P and Q , f is the map $Z \rightarrow Z^{e_p+1} = w$ of the unit disc $U \subset \mathbf{C}$ onto another copy W of it. Since $1, Z, \dots, Z^{e_p}$ provide an \mathcal{O}_W -basis for $f_0(Q_U)$, the value of δ on a local generator of $\mathcal{L} \otimes \mathcal{L}$ is given by

$$\det(\tau(Z^{i+j})), \quad 0 \leq i, j \leq e = e_p.$$

But

$$\tau(Z^{i+j}) = Z^{i+j} (1 + \zeta^{i+j} + (\zeta^{i+j})^2 + \dots + (\zeta^{i+j})^e),$$

(ζ denoting a primitive $(e+1)$ -st root of unity), hence

$$\begin{aligned} \tau(Z^{i+j}) &= (e+1)Z^{i+j} \quad \text{if } i+j = 0 \quad \text{or } e+1, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Hence $\det(\tau(Z^{i+j}))$ is a (nonzero) constant multiple of $Z^{(e+1)e} = w^e$ as asserted.

If $f^{-1}(Q)$ consists of several points, the situation is a direct sum of those considered above, and δ is indeed as asserted. This proves Theorem (4.1).

(4.5) *Remark.* Let the notation be as above, and let $E(X)$ denote the topological Euler-Poincaré characteristic of X . Then, using the formula $E(X) = \text{number of vertices} - \text{number of edges} + \text{number of faces}$ in any triangulation of X , it is easy to see that $E(X) = rE(Y) - \deg R(Y=\mathbf{P}^1)$. Indeed, choose any triangulation of Y which contains all the images of the ramification points of f as vertices, and lift it to a triangulation of X . Then, while r edges or faces lie over each edge or face of Y , the ramification points reduce the number of vertices over certain vertices of Y , and one gets the formula asserted. Since $E(Y) = 2$, (4.2) yields:

(4.6) **COROLLARY.** $\deg K_X = -E(X) = 2g - 2$, i.e. g is also the topological genus $(1/2)b_1(X)$ of the compact oriented surface X .

§ 5. RIEMANN-ROCH THEOREM (FINAL FORM). SERRE DUALITY

(5.1) **(RIEMANN-ROCH THEOREM).** For any line bundle \mathcal{L} on X ,

$$h^0(\mathcal{L}) - h^0(K \otimes \mathcal{L}^{-1}) = \deg \mathcal{L} - g + 1.$$

Proof: It is enough to prove