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# ON THE GENUS OF GENERALIZED FLAG MANIFOLDS

by Henry H. GLOVER and Guido MISLIN

## INTRODUCTION

Let  $X$  be a nilpotent space of finite type. We denote by  $G(X)$  the genus of  $X$ , i.e. the set of all homotopy types  $Y$  (nilpotent, of finite type) with  $p$ -localizations  $Y_p \simeq X_p$  for all primes  $p$ , (cf. [HMR]). The set  $G(X)$  has been studied extensively in case of  $X$  an  $H$ -space. In particular it is known that for the special unitary group  $SU(n)$  one has

$$|G(SU(n))| \geq \prod_{1 < m < n} (\phi(m!)/2)$$

where  $\phi$  is the Euler function [Z, p. 152]. We are interested in this note in finding non-trivial examples  $X$  with  $G(X) = \{[X]\}$  and we call such spaces *generically rigid*. A large family of such generically rigid spaces is provided by certain generalized flag manifolds. Let

$$G = U(n_1 + n_2 + \dots + n_k)$$

and

$$H = U(n_1) \times U(n_2) \times \dots \times U(n_k),$$

embedded in  $G$  in the obvious way. Then

$$M = M(n_1, n_2, \dots, n_k) = G/H$$

is a generalized flag manifold (generalizing the standard complex flag manifold  $U(n)/T^n$  which corresponds to  $M(1, 1, \dots, 1)$ ). We will show essentially that whenever the homotopy rigidity result for linear actions holds for  $M$  (cf. [L1], [L2], [EL]), then  $M$  is also generically rigid. These two seemingly unrelated rigidity results are tied up by certain results on  $E(X)$  and  $E(X_0)$ , the groups of homotopy classes of self equivalences of  $X$  and  $X_0$ ,  $X_0$  the rationalization of  $X$ .

To make our result more precise, we need some further notation. For

$$M = M(n_1, \dots, n_k) = G/H$$

as above, we write  $N(H)$  for the normalizer of  $H$  in  $G$ . The finite group  $N(H)/H$  acts on  $M$  in an obvious way and it is well known that through that action,  $N(H)/H$  is faithfully represented in  $H^*(M; \mathbf{Q})$ . We can therefore consider  $N(H)/H$  as a subgroup of  $E(M)$  or  $E(M_0)$ . By Theorem 1.1 of [GH2] the canonical map

$$E(M_0) \rightarrow \text{Aut}_{\text{alg}} H^*(M; \mathbf{Q})$$

is a group isomorphism. In particular, the grading automorphisms

$$g(q): H^*(M; \mathbf{Q}) \rightarrow H^*(M; \mathbf{Q})$$

defined by  $g(q)x = q^i x$  for  $x \in H^{2i}(M; \mathbf{Q})$  and  $q \in \mathbf{Q}^*$ , lift to unique self equivalences of  $M_0$  (which we denote also by  $g(q)$ ), and thus

$$\text{Gr}(M_0) = \{g(q) \mid q \in \mathbf{Q}^*\} \subset E(M_0)$$

is a central subgroup isomorphic to  $\mathbf{Q}^*$ .

In all cases of generalized flag manifolds for which  $E(M_0)$  has been computed, the subgroup generated by  $\text{Gr}(M_0)$  and  $N(H)/H$ ,

$$\langle \text{Gr}(M_0), N(H)/H \rangle \subset E(M_0)$$

is all of  $E(M_0)$ . The following conjecture is thus plausible.

*Conjecture C.* Let  $M = M(n_1, n_2, \dots, n_k)$  be a generalized flag manifold. Then

$$E(M_0) = \langle \text{Gr}(M_0), N(H)/H \rangle.$$

A similar conjecture appears in [L1, Conjecture C] but the relationship between the two conjectures is not entirely clear.

The Conjecture C has been verified in the following cases:

- 1)  $n_1 = n_2 = \dots = n_k = 1$  (compare the proof of Thm. 1 in [EL])
- 2)  $n_1 = n_2 = \dots = n_{k-1} = 1, n_k \geq k - 1$  (compare the proof of Theorem 9 in [L1])
- 3)  $n_1 = 2$  and  $k = 2$  (follows from [O])
- 4)  $n_2 > n_1$  and  $k = 2$  ([GH1], [Br])
- 5)  $n_1 = 1, n_2 > 1, n_3 \geq 2n_2^2 - 1$  and  $k = 3$  ([GH2])

The Conjecture C holds therefore for instance for all complex Grassmann manifolds  $G_p(\mathbf{C}^{p+q}) = M(p, q)$  with  $p \neq q$  (since  $M(p, q) \simeq M(q, p)$ ), and for the classical flag manifolds  $U(n)/T^n$ .

Our main theorem may be stated as follows.

**THEOREM.** *Let  $M = M(n_1, \dots, n_k)$  be a generalized flag manifold for which the Conjecture C holds. Then*

$$G(M) = \{[M]\}.$$

*In particular the Grassmann manifolds  $G_p(\mathbb{C}^{p+q})$  for  $p \neq q$  and the flag manifolds  $U(n)/T^n$  are all generically rigid.*

§1. GENUS AND SELF MAPS

Let  $P$  denote a fixed set of primes. Two  $P$ -sequences

$$S_1, S_2: P \rightarrow E(X_0)$$

are called *equivalent*, if there exist maps  $h(0) \in E(X_0)$  and

$$h(p) \in \text{im}(E(X_p) \xrightarrow{\text{can}} E(X_0))$$

such that for all  $p \in P$  one has

$$h(0) S_1(p) = S_2(p) h(p).$$

*Definition 1.1.* We denote by  $P\text{-Seq}(E(X_0))$  the set of equivalence classes of  $P$ -sequences in  $E(X_0)$ .

If  $P$  is a finite set of primes and  $X$  a nilpotent space of finite type, then there is a canonical map

$$\theta: G(X) \rightarrow P\text{-Seq}(E(X_0)).$$

It is defined as follows. Let  $Y \in G(X)$  and  $P = \{p_1, \dots, p_n\}$ . Then the localization  $Y_P$  is a pull-back of maps  $X_{p_i} \xrightarrow{\lambda_i} X_0$ , i.e.  $Y_P \simeq \text{hoinvlim} \{X_{p_i} \xrightarrow{\lambda_i} X_0\}$ . The maps  $\lambda_i$  induce equivalences  $\bar{\lambda}_i \in E(X_0)$  and we put

$$\theta(Y) = \{[\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n]\}.$$

If  $Y_P$  may also be represented by  $\text{hoinvlim} \{X_{p_i} \xrightarrow{\mu_i} X_0\}$ , then there exist maps  $h(0) \in E(X_0)$  and  $\tilde{h}(p_i) \in E(X_{p_i})$ ,  $i \in \{1, \dots, n\}$  rendering the diagrams