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ON THE GENUS OF GENERALIZED FLAG MANIFOLDS

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INTRODUCTION

Let X be a nilpotent space of finite type. We denote by $G(X)$ the genus of X, i.e. the set of all homotopy types Y (nilpotent, of finite type) with p-localizations $Y_p \simeq X_p$ for all primes p, (cf. [HMR]). The set $G(X)$ has been studied extensively in case of X an H-space. In particular it is known that for the special unitary group $SU(n)$ one has

$$|G(SU(n))| \geqslant \prod_{1<m<n} (\phi(m!)/2)$$

where ϕ is the Euler function [Z, p. 152]. We are interested in this note in finding non-trivial examples X with $G(X) = \{[X]\}$ and we call such spaces *generically rigid*. A large family of such generically rigid spaces is provided by certain generalized flag manifolds. Let

$$G = U(n_1 + n_2 + \ldots + n_k)$$

and

$$H = U(n_1) \times U(n_2) \times \ldots \times U(n_k),$$

embedded in G in the obvious way. Then

$$M = M(n_1, n_2, \ldots, n_k) = G/H$$

is a generalized flag manifold (generalizing the standard complex flag manifold $U(n)/T^n$ which corresponds to $M(1, 1, \ldots, 1)$). We will show essentially that whenever the homotopy rigidity result for linear actions holds for M (cf. [L1], [L2], [EL]), then M is also generically rigid. These two seemingly unrelated rigidity results are tied up by certain results on $E(X)$ and $E(X_0)$, the groups of homotopy classes of self equivalences of X and X_0, X_0 the rationalization of X.

To make our result more precise, we need some further notation. For

$$M = M(n_1, \ldots, n_k) = G/H$$

as above, we write $N(H)$ for the normalizer of H in G. The finite group $N(H)/H$ acts on M in an obvious way and it is well known that through that action, $N(H)/H$ is faithfully represented in $H^*(M; \mathbf{Q})$. We can therefore consider $N(H)/H$ as a subgroup of $E(M)$ or $E(M_0)$. By Theorem 1.1 of [GH2] the canonical map

$$E(M_0) \to \mathrm{Aut}_{alg} H^*(M; \mathbf{Q})$$

is a group isomorphism. In particular, the grading automorphisms

$$g(q): H^*(M; \mathbf{Q}) \to H^*(M; \mathbf{Q})$$

defined by $g(q)x = q^i x$ for $x \in H^{2i}(M; \mathbf{Q})$ and $q \in \mathbf{Q}^*$, lift to unique self equivalences of M_0 (which we denote also by $g(q)$), and thus

$$Gr(M_0) = \{g(q) \mid q \in \mathbf{Q}^*\} \subset E(M_0)$$

is a central subgroup isomorphic to $\mathbf{Q}^*$.

In all cases of generalized flag manifolds for which $E(M_0)$ has been computed, the subgroup generated by $Gr(M_0)$ and $N(H)/H$,

$$\langle Gr(M_0), N(H)/H \rangle \subset E(M_0)$$

is all of $E(M_0)$. The following conjecture is thus plausible.

Conjecture C. Let $M = M(n_1, n_2, ..., n_k)$ be a generalized flag manifold. Then

$$E(M_0) = \langle Gr(M_0), N(H)/H \rangle .$$

A similar conjecture appears in [L1, Conjecture C] but the relationship between the two conjectures is not entirely clear.

The Conjecture C has been verified in the following cases:

1) $n_1 = n_2 = ... = n_k = 1$ (compare the proof of Thm. 1 in [EL])

2) $n_1 = n_2 = ... = n_{k-1} = 1, n_k \geqslant k - 1$ (compare the proof of Theorem 9 in [L1])

3) $n_1 = 2$ and $k = 2$ (follows from [O])

4) $n_2 > n_1$ and $k = 2$ ([GH1], [Br])

5) $n_1 = 1, n_2 > 1, n_3 \geqslant 2n_2^2 - 1$ and $k = 3$ ([GH2])

The Conjecture C holds therefore for instance for all complex Grassmann manifolds $G_p(\mathbf{C}^{p+q}) = M(p, q)$ with $p \neq q$ (since $M(p, q) \simeq M(q, p)$), and for the classical flag manifolds $U(n)/T^n$.

Our main theorem may be stated as follows.

THEOREM. *Let* $M = M(n_1, \ldots, n_k)$ *be a generalized flag manifold for which the Conjecture C holds. Then*

$$G(M) = \{[M]\}.$$

In particular the Grassmann manifolds $G_p(\mathbf{C}^{p+q})$ *for* $p \neq q$ *and the flag manifolds* $U(n)/T^n$ *are all generically rigid.*

§1. GENUS AND SELF MAPS

Let P denote a fixed set of primes. Two P-sequences

$$S_1, S_2 : P \to E(X_0)$$

are called *equivalent*, if there exist maps $h(0) \in E(X_0)$ and

$$h(p) \in \operatorname{im}\left(E(X_p) \xrightarrow{\text{can}} E(X_0)\right)$$

such that for all $p \in P$ one has

$$h(0)\, S_1(p) = S_2(p)\, h(p).$$

Definition 1.1. We denote by P-Seq $(E(X_0))$ the set of equivalence classes of P-sequences in $E(X_0)$.

If P is a finite set of primes and X a nilpotent space of finite type, then there is a canonical map

$$\theta : G(X) \to P\text{-Seq}\left(E(X_0)\right).$$

It is defined as follows. Let $Y \in G(X)$ and $P = \{p_1, \ldots, p_n\}$. Then the localization Y_P is a pull-back of maps $X_{p_i} \xrightarrow{\lambda_i} X_0$, i.e. $Y_P \simeq \operatorname{hoinvlim}\{X_{p_i} \xrightarrow{\lambda_i} X_0\}$. The maps λ_i induce equivalences $\bar{\lambda}_i \in E(X_0)$ and we put

$$\theta(Y) = \{[\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n]\}.$$

If Y_P may also be represented by $\operatorname{hoinvlim}\{X_{p_i} \xrightarrow{\mu_i} X_0\}$, then there exist maps $h(0) \in E(X_0)$ and $\tilde{h}(p_i) \in E(X_{p_i})$, $i \in \{1, \ldots, n\}$ rendering the diagrams