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$$T_{m+s+1}(\mathbf{x}, \mathbf{c}) T_{m+s+1}(\mathbf{x}, \mathbf{c}') S_{m+r+1}(\mathbf{x}, \mathbf{e})$$

where  $(\mathbf{c}, \mathbf{c}', \mathbf{e})$  is a fixed  $(0, 1)$ -vector of  $2s+r+3$  elements. But any such vector can be specified by a conjunction of  $2s+r+3$  Boolean literals. Consider the disjunction of the  $r$  such conjunctions and let  $R(\mathbf{c}, \mathbf{c}', \mathbf{e})$  be the polynomial that simulates this Boolean formula at  $(0, 1)$  values. Then clearly

$$Q_m(x) = \sum T(\mathbf{x}, \mathbf{c}) T(\mathbf{x}, \mathbf{c}') S(\mathbf{x}, \mathbf{e}) R(\mathbf{c}, \mathbf{c}', \mathbf{e}),$$

where summation is over  $(\mathbf{c}, \mathbf{c}', \mathbf{e}) \in \{0, 1\}^{2s+r+3}$ .

Let  $A(C, d)$  be the upper bound over every homogeneous polynomial having degree  $d$  and homogeneous program complexity  $C$ , of the minimal size of formula needed to define it in Definition 4. Then the above recursive expression ensures that

$$A(C, d) \leq 3A(3C + d, \lfloor d/2 \rfloor + 1) + O(C).$$

Clearly also  $A(C, 1) \leq 2C$ . Hence if  $d$  is  $p$ -bounded in  $m$  then so is the solution to this recurrence. □

### APPENDIX 2

For completeness we describe here a direct proof of the  $p$ -definability of  $HC$  in the sense of Definition 1.  $HC_{n \times n}(x_{i,j})$  will be the projection under

$$\sigma(u_{k,m}) = 1 \quad \text{for} \quad 1 \leq k, m \leq n$$

of the polynomial in  $\{x_{i,j}, u_{k,m}\}$  defined by

$$Q_{n \times n}(y_{i,j}) \cdot Q_{n \times n}(z_{k,m}) \cdot R^1 \dots R^n$$

with the association  $y_{i,j} \leftrightarrow x_{i,j}$  and  $z_{k,m} \leftrightarrow u_{k,m}$ . Here  $Q_{n \times n}$  is the polynomial that defines the permanent in §3. Its first occurrence with argument  $y$  plays exactly the same role as in the permanent and ensures a cycle cover. The intention of  $z_{k,m}$  is to denote whether the  $k^{\text{th}}$  node in the circuit (starting from node 1, say) is node  $m$ .  $Q_{n \times n}(z_{k,m})$  ensures that this intention is realised. For each  $k$   $R^k$  captures the fact that if  $z_{k,m}$  and  $z_{k+1,r}$  are both 1 then  $y_{m,r}$  must be also. In Boolean notation we require

$$y_{m,r} \vee (\bar{z}_{k,m} \vee \bar{z}_{k+1,r}).$$

As is well known such Boolean formulae can be simulated by polynomials at  $\{0, 1\}$  values (e.g. see Proposition 2 in [13]). To guarantee just one monomial for each cycle we fix  $R^1 = z_{11}$ . □