Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 28 (1982)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FROBENIUS RECIPROCITY AND LIE GROUP REPRESENTATIONS ON $\bar{\delta}$ COHOMOLOGY SPACES

Autor: Williams, Floyd L.

Kapitel: 2. Induction and reciprocity

DOI: https://doi.org/10.5169/seals-52231

2. Induction and reciprocity

The notion of induced representations for finite groups was introduced in 1898 by G. Frobenius in the paper [37]. In the same paper Frobenius established what is now called the Frobenius reciprocity relation. We recall his basic construction which is fundamental in the entire theory of group representations. [1])

Let G be a finite group and let P be a subgroup of G. Let π be a representation of G on a finite dimensional vector space V. That is $\pi: G \to GL(V)$ is a homomorphism of G into the group of non-singular endomorphisms of V. We shall also refer to V as a (left) G module. By restriction V is also a P module. Conversely there is a functor I which converts P modules to G modules: Given a P module W the G module IW is defined to be the space of functions $f: G \to W$ such that $f(ap) = p^{-1} \cdot f(a)$ for every (a, p) in $G \times P$. The action of G on IW is defined by

$$(a \cdot f)(x) = f(a^{-1}x)$$

for (f, a, x) in $(I\,W) \times G \times G$. $I\,W$ is called the G module *induced* by the P module W. Induction and restriction are related in the following way.

Theorem 2.1 (Frobenius reciprocity relation, 1898). *If W is a P module and if V is a G module then*

$$\mathrm{Hom}_G(V, I\,W) = \mathrm{Hom}_P(V, W).$$

We wish to consider extensions or analogues of this relation in a wider context. For this it is most convenient first of all to re-describe the G module $I\,W$. The following "geometric" interpretation of $I\,W$ is well-known. Consider the right action of P on $G \times W$ given by

$$(a, w) \cdot p = (ap, p^{-1}w)$$

for (a, p, w) in $G \times P \times W$. Let

$$E_W = \text{orbit space } (G \times W)/P = G \times_P W. \tag{2.2}$$

Let $\gamma: E_W \to G/P$ be the canonical (well-defined) map $[a, w] \to aP$, where $[a, w]$ is the orbit of $(a, w) \in G \times W$. For each $a \in G$ the map $w \to [a, w]$ of W to $\gamma^{-1}\{aP\}$ is a bijection. That is we may identify W as the fibre over each point of

---

[1]) For the theory of induced representations of locally compact groups see G. Mackey [55], [56].

G/P. G acts naturally on E_W and G/P on the left. γ is an equivariant map. Let $\Gamma(E_W)$ be the space of sections of E_W. That is $s \in \Gamma(E_W)$ is a map from G/P to E_W satisfying $\gamma \circ s = 1$; hence s maps each point to the fibre over it. $\Gamma(E_W)$ is a left G module:

$$(a \cdot s)(x) = a \cdot s(a^{-1} \cdot x) \tag{2.3}$$

for (a, s, x) in $G \times \Gamma(E_W) \times G/P$. Moreover

PROPOSITION 2.4. *There is a natural G module isomorphism $s \to f^s$ of $\Gamma(E_W)$ onto $I\,W$ such that for every a in G, $s(aP) = [a, f^s(a)]$. Hence by Theorem 2.1*

$$\mathrm{Hom}_G(V, \Gamma(E_W)) = \mathrm{Hom}_P(V, W). \tag{2.5}$$

This sets the stage for a possible extension of Frobenius. Namely, following Bott, we consider the following data. G is a complex Lie group, P is a closed complex Lie subgroup (thus the injection $P \to G$ is holomorphic), and W is a finite dimensional holomorphic P module (i.e. for each w in W and f in the complex dual space of W the map $p \to f(p \cdot w)$ of P to the complex numbers is holomorphic). We define E_W exactly as above. Then E_W has the structure of a holomorphic vector bundle over the complex manifold G/P. Let $\Gamma(E_W)$ now denote the space of C^∞ sections with the G module structure given by (2.3) and let $\Gamma_{\mathrm{hol}}(E_W)$ denote the G stable subspace of holomorphic sections. Since all of our data is now holomorphic the most natural question to ask, considering (2.5), is: When is it true that

$$\mathrm{Hom}_G(V, \Gamma_{\mathrm{hol}}(E_W)) = \mathrm{Hom}_P(V, W) \tag{2.6}$$

for a holomorphic G module V? (2.6) would then represent an exact holomorphic analogue of Frobenius reciprocity. It turns out that (2.6) is valid if the space G/P is sufficiently nice. For example suppose that G/P is a compact simply connected Kahler manifold. Group theoretically this means that G is a connected complex semisimple Lie group and P is a parabolic subgroup. Then it is due to Bott [12] that (2.6) is valid. In fact in [12] Bott proves considerably more: Let SE_W be the sheaf of germs of local holomorphic sections of E_W and let $H^*(G/P, SE_W)$ be the cohomology of G/P with coefficients in SE_W. Then we have

THEOREM 2.7 (R. Bott, 1957). *Suppose G is a connected complex semisimple Lie group and P is a parabolic subgroup of G. Let $\mathfrak{p}$ be the Lie algebra of P and let V, W be finite dimensional holomorphic G and P modules respectively. Then*

(2.8) $$\mathrm{Hom}_G(V, H^j(G/P, SE_W)) = H^j(p, p \cap \bar{p}, \mathrm{Hom}(V, W))$$

for each $j \geqslant 0$.

The bar – denotes conjugation of G with respect to a maximal compact subgroup K of G and the right hand side of (2.8) is the *relative* Lie algebra cohomology of p (in the sense of Hochschild, Serre [44]). Here $H^j(G/P, SE_W)$ [1]) has the G module structure induced by the left action of G on E_W and Hom (V, W) has the p module structure defined by

(2.9) $$(x \cdot \phi)(v) = -\phi(x \cdot v) + x \cdot \phi(v)$$

for (x, ϕ, v) in $p \times \mathrm{Hom}(V, W) \times V$.

Remarks. (i) For $j = 0$, $H^0(p, p \cap \bar{p}, \mathrm{Hom}(V, W))$ is independent of the subalgebra $p \cap \bar{p}$ of p and has the value $\mathrm{Hom}(V, W)^P$ (the space of invariants) which is precisely $\mathrm{Hom}_p(V, W) = \mathrm{Hom}_P(V, W)$ by (2.9) (P is connected). Also $H^0(G/P, SE_W)$ is precisely $\Gamma_{\mathrm{hol}}(E_W)$. Thus taking $j = 0$ in (2.8) we get

$$\mathrm{Hom}_G(V, \Gamma_{\mathrm{hol}}(E_W)) = \mathrm{Hom}_P(V, W)$$

which is (2.6). This shows that (2.8) represents a rather remarkable extension of Frobenius reciprocity to higher cohomology. Here the induction functor is $I: W \to H^*(G/P, SE_W)$.

(ii) As shown by Bott (2.8) is valid, more generally, for C-spaces G/P in the sense of Wang [90]. The latter need not be Kahler, as we have assumed for our purposes.

The functor I in remark (i) can be explicated by the use of differential forms: Let $\Lambda^{0,j}(G/P, E_W)$ denote the space of E_W valued C^∞ differential forms on G/P of pure type $(0, j)$. That is

$$\omega \in \Lambda^{0,j}(G/P, E_W)$$

assigns to each $x \in G/P$ a skew-symmetric j linear map

$$\omega_x: T_x(G/P)^{\mathbf{C}} \times \ldots \times T_x(G/P)^{\mathbf{C}} \to (E_W)_x = \gamma^{-1}\{x\}$$

on the complexified tangent space $T_x(G/P)^{\mathbf{C}}$ of G/P at x to the fiber $(E_W)_x$ over x such that (a) given smooth vector fields $X_1, \ldots, X_j$ on G/P the map

$$\omega(X_1, \ldots, X_j): x \to \omega_x(X_{1_x}, \ldots, X_{j_x})$$

is C^∞—i.e. it belongs to $\Gamma(E_W)$ and (b) for each real number θ,

$$\omega(U_\theta X_1, \ldots, U_\theta X_j) = e^{-\sqrt{-1}\, j\theta}\, \omega(X_1, \ldots, X_j)$$

[1]) Since G/P is compact $H^j(G/P, SE_W)$ is known to be finite-dimensional.

where

$$U_\theta X_l = \cos\theta\, X_l + \sin\theta\, JX_l$$

and J is the complex structure tensor on G/P. Let $\bar\partial\colon \Lambda^{0,j} \to \Lambda^{0,j+1}$ denote, as usual, the Cauchy-Riemann operator so that $\bar\partial^2 = 0$. If f is a C^∞ function on G/P and X is a C^∞ vector field on G/P then

$$(\bar\partial f)(X) = \frac{1}{2}[Xf + \sqrt{-1}(JX)f]. \tag{2.10}$$

Since $\bar\partial^2 = 0$ let $H^{0,j}_{\bar\partial}(G/P, E_W)$ denote the corresponding $\bar\partial$ cohomology:

$$H^{0,j}_{\bar\partial}(G/P, E_W) = \frac{\ker \bar\partial\colon \Lambda^{0,j}(G/P, E_W) \to \Lambda^{0,j+1}(G/P, E_W)}{\bar\partial\Lambda^{0,j-1}(G/P, E_W)}. \tag{2.11}$$

By Dolbeault's theorem [35]

$$H^j(G/P, SE_W) = H^{0,j}_{\bar\partial}(G/P, E_W). \tag{2.12}$$

The induced action of G on $H^{0,j}_{\bar\partial}(G/P, E_W)$ is given explicitly as follows. First G acts on $\Lambda^{0,j}(G/P, E_W)$ by

$$(a\cdot\omega)_x(L_1, ..., L_j) = a\cdot\omega_{a^{-1}x}\left(dl_{a^{-1}x}(L_1), ..., dl_{a^{-1}x}(L_j)\right) \tag{2.13}$$

where

$$(a, \omega, x) \in G \times \Lambda^{0,j}(G/P, E_W) \times G/P,$$

each $L_l \in T_x(G/P)^{\mathbf{C}}$ and dl_{a_x} is the derivative of left translation $l_a\colon G/P \to G/P$ on G/P at x. Note that (2.13) generalizes the action of G on

$$\Gamma(E_W) = \Lambda^{0,0}(G/P, E_W)$$

given in (2.3). Because left translation is holomorphic the diagram

$$\begin{array}{ccc} \Lambda^{0,j}(G/P, E_W) & \xrightarrow{\bar\partial} & \Lambda^{0,j+1}(G/P, E_W) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ \Lambda^{0,j}(G/P, E_W) & \xrightarrow{\bar\partial} & \Lambda^{0,j+1}(G/P, E_W) \end{array}$$

is commutative for each a in G. Thus (2.13) induces a well-defined action of G on $H^{0,j}_{\bar\partial}(G/P, E_W)$. We may now write (2.8) as

$$\mathrm{Hom}_G(V, H^{0,j}_{\bar\partial}(G/P, E_W)) = H^j(p, p\cap\bar p, \mathrm{Hom}(V, W)). \tag{2.14}$$

Now assume that W is in fact irreducible. The parabolic subalgebra p has a decomposition $p = (p\cap\bar{p}) \oplus n$ into a reductive part $p \cap \bar{p}$ and a nilpotent part $n =$ an ideal in p. By general principles

$$H^j(p, p\cap\bar{p}, \mathrm{Hom}(V, W)) = H^j(n, \mathrm{Hom}(V, W))^{p\cap\bar{p}}$$
$$= H^j(n, V^*\otimes W)^{p\cap\bar{p}} = \left(H^j(n\, V^*) \otimes W\right)^{p\cap\bar{p}}.$$

The last statement of equality follows by the irreducibility of W since by Lie's theorem, W is a trivial n module. Now

$$\left(H^j(n, V^*) \otimes W\right)^{p\cap\bar{p}} = \mathrm{Hom}_{p\cap\bar{p}}\left(W^*, H^j(n, V^*)\right).$$

From (2.14) we obtain (see [50]).

THEOREM 2.15 (Bott-Kostant reciprocity, 1960). *Let G, P be as in Theorem 2.7, let n be the nilradical of the parabolic subalgebra p, and let W be a finite dimensional irreducible holomorphic P module. Then for any finite dimensional holomorphic G module V we have*

(2.16) $$\mathrm{Hom}_G\left(V, H^{0,j}_{\bar{\partial}}(G/P, E_W)\right) = \mathrm{Hom}_{p\cap\bar{p}}\left(W^*, H^j(n, V^*)\right).$$

Again $p \cap \bar{p}$ is the reductive part of p where the bar denotes conjugation of $G = K^{\mathbf{C}}$ with respect to a maximal compact subgroup K. We refer to (2.16) as "the debut of n cohomology"! Since 1960 it has played some rather important roles in both finite dimensional and infinite dimensional representation theory. There is an equivalent version of (2.16): The G module structure on $H^{0,j}_{\bar{\partial}}(G/P, E_W)$ induced by (2.13) may be restricted to K. Let $\hat{K}$ denote, as usual, the equivalence classes of the irreducible unitary representations of K and let V_π be the representation space of $\pi \in \hat{K}$. Then we have (again for W irreducible).

THEOREM 2.17 (B. Kostant). *The decomposition of $H^{0,j}_{\bar{\partial}}(G/P, E_W)$ as a K module is*

(2.18) $$H^{0,j}_{\bar{\partial}}(G/P, E_W) = \sum_{\pi\in\hat{K}} V_\pi \otimes \mathrm{Hom}_{p\cap\bar{p}}\left(W^*, H^j(n, V_\pi^*)\right)$$
$$= \sum_{\pi\in\hat{K}} V_\pi^* \otimes \mathrm{Hom}_{p\cap\bar{p}}\left(W^*, H^j(n, V_\pi)\right).$$

In the direct sum on the right hand side the action of K on a summand is $\pi \otimes 1$ or $\pi^* \otimes 1$ in the second equation.

From (2.18) (or from (2.16)) we see that the multiplicity of an irreducible K module V_π in $H^{0,j}_{\bar{\partial}}(G/P, E_W)$ is governed precisely by the n cohomology

$H^j(n, V_\pi^*)$. Here, by analytic continuation, we consider V_π also as a representation of the complex Lie algebra of G. Its n module structure is the restriction thereof to n.

Remarks. (i) In contrast to remark (ii) made earlier, following Theorem 2.7, Theorems (2.15) and (2.17) do require that G/P should be Kahler.

(ii) One knows that K acts transitively on G/P so that G/P is diffeomorphic to $K/K \cap P$.

Now Kostant in [50] has computed the Lie algebra cohomology groups $H^j(n, V_\pi^*)$. Two outstanding consequences of his results, among others, which we shall briefly discuss are (a) Weyl's character formula and (b) Bott's generalized Borel-Weil theorem. Suppose more generally that g is any complex semisimple Lie algebra (for example g could be the Lie algebra of G above). Let $h \subset g$ be a Cartan subalgebra of g, let Δ be the set of non-zero roots of (g, h), and let Δ^+ be a choice of positive roots. The equivalence classes of finite dimensional irreducible representations of g (over the complex numbers) correspond univalently to linear functionals Λ on h which satisfy the condition that $2\dfrac{(\Lambda, \alpha)}{(\alpha, \alpha)}$ is a non-negative integer for each α in Δ^+. That is Λ is Δ^+ *dominant integral*; $(\, , \,)$ denotes the Killing form on g. This is Cartan's highest weight theory alluded to in the introduction. Let π_Δ be a finite dimensional irreducible representation of g with corresponding highest weight $\Lambda \in h^*$. Its *character* $X_\Lambda: h \to \mathbf{C}$ is defined to be the function $H \to$ trace $\exp \pi_\Lambda(H)$, $H \in h$. This definition is independent of the choice of Cartan subalgebra h since any two are conjugate. We consider the special "minimal" parabolic subalgebra $p \subset g$ whose nilradical is

$$n = \sum_{\alpha \in \Delta^+} g_\alpha \tag{2.19}$$

and whose reductive part is h where g_α is the root space of $\alpha \in \Delta$. That is p is just the Borel subalgebra $h + n$. Let V_Λ denote the representation space of π_Λ. Then by restriction to n we again form the Lie algebra cohomology groups $H^j(n, V_\Lambda)$. Let θ denote the adjoint representation of h on Λn^*. Then $\theta \otimes \pi_\Lambda$ defines a representation of h on the cochain complex $\Lambda n^* \otimes V_\Lambda$. This h action commutes with the coboundary operator and therefore passes to cohomology. Applying the Euler-Poincaré principle one gets

$$\sum_{j=0}^{\dim n} (-1)^j \text{ trace } \exp \theta \otimes \pi_\Lambda(H) \Big|_{\Lambda^j n^* \otimes V_\Lambda} = \sum_{j=0}^{\dim n} (-1)^j \text{ trace } \exp \theta \otimes \pi_\Lambda(H) \Big|_{H^j(n, V_\Lambda)} \tag{2.20}$$

for each H in h. One evaluates the left hand side of (2.20) by general principles and the right hand side using Kostant's main theorem, Theorem 5.14 of [50]. Actually Theorem 5.14 of [50] gives the h_1 module structure of $H^j(n_1, V_\Lambda)$ for an *arbitrary* parabolic $p_1 = h_1 + n_1$ of g with reductive and nilpotent parts h_1, n_1 respectively. For the derivation of Weyl's formula only the simplest case $p_1 = p = h + n$ is needed, where n is given in (2.19). Thus we shall state only a special case of Kostant's result.

THEOREM 2.21 (B. Kostant, 1960). *The decomposition of* $H^j(n, V_\Lambda)$ *as a* h *module is*

$$H^j(n, V_\Lambda) = \sum V_{\Lambda,\sigma},$$

$$\sigma \in \text{Weyl group } \mathscr{W} \text{ of } (g, h) \text{ such that } l(\sigma) = j,$$

where each summand $V_{\Lambda,\sigma}$ *in the direct sum is one-dimensional and* $H \in h$ *acts on* $V_{\Lambda,\sigma}$ *by the scalar* $[\sigma(\Lambda+\delta)-\delta](H)$.

Here by definition $2\delta = \sum_{\alpha\in\Delta^+} \alpha$ and $l(\sigma)$ (the *length* of σ) is the cardinality of the set $\Delta^+ \cap \sigma(-\Delta^+)$. From the remarks following (2.20) and the knowledge of n cohomology given by Theorem 2.21 one derives Weyl's famous character formula [93]:

THEOREM 2.22 (H. Weyl, 1926). *For* $H \in h$

$$X_\Lambda(H) = \frac{\sum_{\sigma\in\mathscr{W}} (\det\sigma)\, e^{[\sigma(\Lambda+\delta)](H)}}{\prod_{\alpha\in\Delta^+} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}.$$

The denominator is also given by the sum $\sum_{\sigma\in\mathscr{W}} (\det\sigma)\, e^{(\sigma\delta)(H)}$ (this fact can be proved too using n cohomology) and $\det\sigma = (-1)^{l(\sigma)}$. As a corollary of Theorem 2.22 one obtains Weyl's formula for the dimension of the irreducible module V_Λ in terms of its highest weight Λ. The result is

(2.23)
$$\dim V_\Lambda = \frac{\prod_{\alpha\in\Delta^+} (\Lambda+\delta, \alpha)}{\prod_{\alpha\in\Delta^+} (\delta, \alpha)}.$$

Kostant's result on n cohomology can also be used to derive the generalized Borel-Weil theorem. Here one may apply formula (2.18) decisively. Let g now denote the Lie algebra of G. Extend a maximal abelian subalgebra of the Lie algebra of K to a Cartan subalgebra h of g. Again let $\Delta^+ \subset \Delta$ be a choice of positive roots where Δ is the set of non-zero roots of (g, h) and let $2\delta = \sum_{\alpha\in\Delta^+} \alpha$.

We choose the parabolic P such that its Lie algebra p contains the Borel subalgebra $h + \sum_{\alpha \in \Delta^+} g_{-\alpha} \cdot h$ is also a Cartan subalgebra of the reductive Lie algebra $p \cap \bar{p}$ so that we have the decompositions

$$p = (p \cap \bar{p}) \oplus n, \qquad p \cap \bar{p} = h + \sum_{\alpha \in \Delta(p \cap \bar{p})} g_\alpha$$

(2.24)

$$n = \sum_{\alpha \in \Delta^+ - \Delta(p \cap \bar{p})} g_{-\alpha}$$

where $\Delta(p \cap \bar{p})$ is the set of roots of $(p \cap \bar{p}, h)$.

Let W be an irreducible holomorphic P module. Then W is an irreducible $p \cap \bar{p}$ module thereby such that $n \cdot W = 0$. We let Λ denote its highest weight relative to the positive system $\Delta^+ \cap \Delta(p \cap \bar{p})$ for $p \cap \bar{p}$. Applying Kostant's n cohomology theorem to (2.18) one obtains (see [12], [50]).

THEOREM 2.25 (R. Bott, 1957). *The spaces* $H^{0,j}_{\bar{\partial}}(G/P, E_W)$ *vanish for all but at most one* j. *If*

$$H^{0,j_0}_{\bar{\partial}}(G/P, E_W) \neq 0$$

then $H^{0,j_0}_{\bar{\partial}}(G/P, E_W)$ *is an irreducible* K *module.*

More precisely we have the following. Let Λ be the highest weight of W (as above) relative to the positive roots in the reductive part of P. If $(\Lambda + \delta, \alpha) = 0$ for some α in Δ then $H^{0,j}_{\bar{\partial}}(G/P, E_W) = 0$ for every j. If $(\Lambda + \delta, \alpha) \neq 0$ for each α in Δ (i.e. $\Lambda + \delta$ is *regular*) there is a unique element σ in the Weyl group of (g, h) such that $(\sigma(\Lambda + \delta), \alpha) > 0$ for every $\alpha \in \Delta^+$. Then $H^{0,j}_{\bar{\partial}}(G/P, E_W) = 0$ for $j \neq l(\sigma)$ where again $l(\sigma)$ is the length of σ (see remarks following Theorem 2.21). Moreover $H^{0,l(\sigma)}_{\bar{\partial}}(G/P, E_W)$ is an irreducible K module (= an irreducible g module since g is the complexification of the Lie algebra of K) with highest weight $\sigma(\Lambda + \delta) - \delta$ relative to Δ^+.

Remarks. (i) By definition of σ it follows that

$$\sigma^{-1}\Delta^- \cap \Delta^+ = \{\alpha \in \Delta^+ \mid (\Lambda + \delta, \alpha) < 0\}.$$

Also since Λ is a highest weight $(\Lambda, \alpha) \geqslant 0$ for

$$\alpha \in \Delta^+ \cap \Delta(p \cap \bar{p}) \Rightarrow (\Lambda + \delta, \alpha) > 0$$

for

$$\alpha \in \Delta^+ \cap \Delta(p \cap \bar{p}).$$

Hence

$$\{\alpha \in \Delta^+ \mid (\Lambda + \delta, \alpha) < 0\}$$
$$= \{\alpha \in \Delta^+ - (\Delta^+ \cap \Delta(p \cap \bar{p})) \mid (\Lambda + \delta, \alpha) < 0\}$$

so that $l(\sigma)$ in Theorem 2.25 has the value

$$|\{\alpha \in \Delta^+ - (\Delta^+ \cap \Delta(p \cap \bar{p})) \mid (\Lambda + \delta, \alpha) < 0\}|\ [1]\,.$$

$$\Delta^+ - \Delta^+ \cap \Delta(p \cap \bar{p})$$

is the set of roots in the nilradical of the "opposite" parabolic $\bar{p}$. Since

$$(\sigma(\Lambda + \delta), \sigma\alpha) = (\Lambda + \delta, \alpha) > 0$$

for $\alpha \in \Delta^+ \cap \Delta(p \cap \bar{p})$ (as we have just seen) we also conclude that the Weyl group element σ in Theorem 2.25 satisfies

$$\Delta^- \cap \Delta(p \cap \bar{p}) \subset \sigma^{-1} \Delta^- .$$

(ii) The irreducible holomorphic P modules W in the statement of Theorem 2.25 can be obtained as follows. Start with an arbitrary irreducible representation π of $P \cap K$ on a complex vector space W. Since $p \cap \bar{p}$ is the complexification of the Lie algebra of $P \cap K$, π defines a unique irreducible representation π on p such that $\pi(n) = 0$. This infinitesimal representation can be "integrated" to a representation of P since P and $P \cap K$ have the same fundamental groups. Thus every irreducible representation π of $P \cap K$ extends uniquely to an irreducible holomorphic representation of P. The highest weight Λ of π is integral and $\Delta^+ \cap \Delta(p \cap \bar{p})$ dominant. Conversely if G is simply connected, any integral $\Lambda \in h^*$ which is $\Delta^+ \cap \Delta(p \cap \bar{p})$ dominant is the highest weight of irreducible representation of $P \cap K$ and hence is the highest weight of an irreducible holomorphic representation of P.

(iii) Suppose in particular G is simply connected, p is chosen to be

$$h + \sum_{\alpha \in \Delta^+} g_{-\alpha}\,,$$

and that Λ is Δ^+ dominant integral. Then in Theorem 2.25 $\sigma = 1$ so that the irreducible K, G or g module with highest weight Λ is given by $H^{0,0}_{\bar{\partial}}(G/P, E_W)$ = space of holomorphic sections of the line bundle E_W. Indeed $\dim_{\mathbf{C}} W = 1$ since in this case $P \cap K$ is abelian. This gives the geometric realization of V_Λ [11].

[1]) $|S|$ denotes the cardinality of a set S.