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unless V_π^∞ has a specific infinitesimal character (this means that on V_π^∞ the center of the universal enveloping algebra Ug of g must act by a specific scalar). In Harish-Chandra's notation [20] this character is $\chi_{-\mu-\delta}$, where again $2\delta = \sum_{\alpha \in \Delta^+} \alpha$; here $\mu \in h^*$ is integral or, more generally, μ is real-valued on roots and defines a character of H (see remarks following Theorem 3.11). On the other hand, from the harmonic analysis of G it is known that only finitely many irreducible unitary equivalence classes can have a fixed infinitesimal character and that moreover if $F \subset \hat{G}$ is a finite set which is disjoint from the classes of discrete series, then the Plancherel measure must vanish on F. Thus from these observations one concludes from (3.15) that only discrete series modules (π, V_π) can occur in the direct integral decomposition given in (3.12) and since the (π, V_π) occur discretely we obtain (cf. Corollary 3.23 of [82]) the following refinement of (3.12).

THEOREM 3.16 (Frobenius-Schmid reciprocity, 1975). *As G modules*

$$H^{0,j}_{\bar{\partial},2}(G/H, \mathscr{L}_\Lambda) = \sum_{\substack{(\pi, V) = \\ \text{discrete class}}} V_\pi^* \otimes H^j(n, V_\pi^\infty)_{e^{-\Lambda}}$$

This is the non-compact analogue of (2.18) (where the contragradient W^* of the inducing module W there is replaced by the contragradient $e^{-\Lambda}$ of the inducing character e^Λ). Theorem 3.16 preceeds and implies (with the knowledge of n cohomology, as in the compact case) Theorem 3.11.

4. REMARKS ON THE NILPOTENT CASE: POLARIZATIONS AND HARMONIC INDUCTION

The Frobenius reciprocity in higher cohomology discussed in the two preceding sections extends to a non-semisimple Lie group context as well. Moreover consequent analogues of the Kostant-Langlands conjecture have been proved. Most recently (within the past few months) remarkable and complete results along these lines have been obtained (independently) for simply connected nilpotent Lie groups by J. Rosenberg [74] and R. Penney [69]. Their results are preceded by results of H. Moscovici and A. Verona [59]; also see [15], [58], [62], [67], [68], [75]. In this regard one of the central notions to consider is that of a *polarization*. It is defined as follows. Let g be a real Lie algebra, let $\Lambda \in g^*$

be a linear functional, and let B_Λ be the skew symmetric bilinear form on g defined by

$$B_\Lambda(x, y) = \Lambda([x, y]), x, y \in g.$$

A (complex) polarization of g at Λ is a complex subalgebra p of $g^{\mathbf{C}}$ which is maximally isotropic relative to B_Λ and which has the further property that $p + \bar{p}$ is also a complex subalgebra of $g^{\mathbf{C}}$. The bar denotes conjugation of $g^{\mathbf{C}}$ relative to the real form g. Λ defines a hermitian sequilinear form H_Λ on

$$p: H_\Lambda(x, y) = \sqrt{-1}\,\Lambda([x, \bar{y}]) = \sqrt{-1}\,B_\Lambda(x, \bar{y}), x, y \in p.$$

Define

(4.1) $$q(p, \Lambda) = \dim_{\mathbf{C}} \frac{(p \cap \bar{p})}{\text{radical of } B_\Lambda} + \text{number of negative signs in the signature of } H_\Lambda \text{ on } \frac{p}{p \cap \bar{p}}.$$

This important invariant of the polarization is called its *negativity index*.

Let G now denote a connected, simply connected nilpotent Lie group with Lie algebra g and let p be a complex polarization of g at $\Lambda \in g^*$. Let d and e denote the subalgebras of g defined by

$$d = p \cap g, e = (p+\bar{p}) \cap g;$$

hence

$$d^{\mathbf{C}} = p \cap \bar{p}, e^{\mathbf{C}} = p + \bar{p}.$$

Let $D, E \subset G$ be the corresponding connected Lie subgroups of G. One knows that D and E are closed (and simply connected) in G and the quotient $X = E\backslash D$ has a unique E invariant complex structure such that $p/d^{\mathbf{C}}$ is the anti-holomorphic tangent space at the origin. This is proved by Kostant in [53] for example. However X may not necessarily admit an E invariant hermitian metric. A sufficient condition for the existence of the latter is that $p \cap \bar{p}$ should be an ideal in p (for then the image of Ad (D) in Hom (e/d) is compact). Polarizations satisfying this sufficient condition are called *relatively ideal.* Now let X_Λ be the unique unitary character of D with differential $2\pi\sqrt{-1}\,\Lambda|_d$ and let $\mathscr{L}_\Lambda \to X$ be the corresponding induced C^∞ line bundle over X. Then a priori $\mathscr{L}_\Lambda$ admits an E invariant hermitian metric and a holomorphic structure. Thus if p is relatively ideal, which we now assume, so that X also admits an E invariant hermitian metric, the pair $(\mathscr{L}_\Lambda, X)$ is a hermitian bundle and the corresponding L^2-

cohomology groups $H^{0,j}_{\bar\partial,2,p,E}(X,\mathscr{L}_\Lambda)$ can therefore be defined; see 3.10 and [63]. This time we denote their dependence on p. Since the above metrics are E invariant these groups (= Hilbert spaces) carry a natural unitary representation of E (as in section 3) which we denote by $\pi^{0,j}_{\Lambda,p,E}$. Now form the induced representation of G in the sense of G. Mackey [55]:

$$\pi^j(\Lambda,p,G) \overset{def}{=} \operatorname*{ind}_{E\uparrow G} \pi^{0,j}_{\Lambda,p,E} \tag{4.2}$$

$\pi^j(\Lambda,p,G)$ is the j-th *harmonically* induced representation of G associated to the polarization p at Λ in the sense of Moscovici and Verona [59].

Now Theorem 3.5 of section 3 gives the "Harish-Chandra correspondence" $\Lambda_1 \to \pi_{\Lambda_1}$ where Λ_1 is an integral linear form on a Cartan subalgebra such that $\Lambda_1+\delta$ is regular and π_{Λ_1} is the corresponding discrete series representation. Similarly there is for connected, simply connected nilpotent Lie groups G_1 the well known *Kirillov correspondence* $\Lambda_1 \to \pi_{\Lambda_1}$ [1]) $\in \hat{G}_1$ for $\Lambda_1 \in$ dual space of the Lie algebra g_1 of G_1, where in fact the whole unitary dual space $\hat{G}_1$ of G_1 is parametrized by the orbits in g_1^* under the contragredient action of the adjoint representation of G_1; see [49], [9], [72]. In terms of the Kirillov correspondence and harmonically induced representations we shall discuss another version of Frobenius reciprocity.

Recall the *formal harmonic spaces* $\mathscr{H}^j(\pi)$ of Schmid which appeared in equation (3.12) and defined thereafter. One may define similarly the j-th formal harmonic spaces $\mathscr{H}^j(\pi,p)$ of $\pi\in\hat{E}$ associated to the polarization p; see page 67 of [59]. In Lemma 4 of [59], or Theorem 10 of [68], Moscovici, Verona, and Penney prove

THEOREM 4.3 (1978). *Let p be a relatively ideal polarization at $\Lambda\in g^*$ as above and let $\pi_{\pm\Lambda|e}\in\hat{E}$ be the Kirillov representations of E corresponding to $\Lambda|_e\in e^*$. Then there is a Hilbert space isomorphism*

$$\begin{aligned} &H^{0,j}_{\bar\partial,2,p,E}(X,\mathscr{L}_\Lambda)\\ &= (\text{representation space of } \pi_{\Lambda|e})\otimes\mathscr{H}^j(\pi_{-\Lambda|e},p) \end{aligned} \tag{4.4}$$

such that $\pi^{0,j}_{\Lambda,p,E}=\pi_{\Lambda|e}\otimes 1$.

In other words the multiplicity of the Kirillov representation $\pi_{\Lambda|e}$ in the representation $\pi^{0,j}_{\Lambda,p,E}$ on $\bar\partial$-cohomology is given by the dimension of the formal harmonic space $\mathscr{H}^j(\pi|_{-\Lambda|e},p)$. This result is rather similar to Theorem 3.16

[1]) π_{Λ_1} is the representation of G_1 induced by a unitary character corresponding to Λ_1 of a closed subgroup corresponding to a *real* polarization at Λ_1.

since by (3.15) the Lie algebra cohomology space in Theorem 3.16 is a formal harmonic space. It is even true as a matter of fact that under reasonable conditions the formal harmonic space associated to a polarization coincides with a Lie algebra cohomology space; see Penney's Theorem 2 in [68]. The latter cohomology spaces have the form $H^j(p \cap \operatorname{Ker} \Lambda, \pi^{\infty}_{-\Lambda})$ where π^{∞} is the space of C^{∞} vectors in a representation π and Λ is considered also as a linear functional on $g^{\mathbf{C}}$. By very clever means these spaces are shown to vanish for all j except j = the negativity index $q(p, \Lambda)$ of the polarization (see 4.1). Moreover $H^{q(p, \Lambda)}(p \cap \operatorname{Ker} \Lambda, \pi^{\infty}_{-\Lambda})$ is one-dimensional; see Rosenberg's Theorem 2.4 in [74]; also see Penney [69]. With these remarks in mind an application of Theorem 4.3 gives

THEOREM 4.5 (J. Rosenberg–R. Penney 1979). *Let G be a connected, simply connected nilpotent Lie group and let p be a relatively ideal complex polarization at $\Lambda \in g^*$, g = Lie algebra of G. Let $\pi^j(\Lambda, p, G)$ be the j-th harmonically induced representation defined in (4.2). Then $\pi^j(\Lambda, p, G)$ vanishes for $j \neq$ the negativity index $q(p, \Lambda)$ (see (4.1)). Moreover $\pi^{q(p, \Lambda)}(\Lambda, p, G)$ is irreducible and unitarily equivalent to the Kirillov representation π_{Λ}.*

Theorem 4.5 is clearly analogous to Theorem 3.11 and thus it represents the confirmation of a version of the Kostant-Langlands conjecture for nilpotent Lie groups. One may add that as a matter of fact the distinguished integer q_{Λ} in (3.9) is indeed the negativity index of a complex polarization-namely the polarization is a Borel subalgebra at a regular point.

5. FURTHER NOTES

1. We have pointed out earlier that in addition to Schmid's thesis work, early efforts towards proving the Kostant-Langlands conjecture were made by Narasimhan and Okamoto. The latter authors considered the special case when G/K admits a G invariant complex structure [1]). They constructed unitary representations $\pi_{\Lambda}^{0,j}$ of G on L_2-cohomology spaces associated to holomorphic vector bundles E_{Λ} over G/K induced by an irreducible unitary representation of K with highest weight Λ; compare remarks following (3.10). The $\pi_{\Lambda}^{0,j}$ are shown to be subject to an important *alternating sum formula* which, roughly stated, says that

(5.1) $$\sum_{j=0}^{n} (-1)^j \text{ character of } \pi_{\Lambda}^{0,j} = (-1)^{q_{\Lambda}} \text{ character of } \pi_{\Lambda}^{*}$$

[1]) Here G, K are as in section 3.